SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION AND CONVOLUTION TECHNIQUES TO UNIVALENT FUNCTIONS THEORY
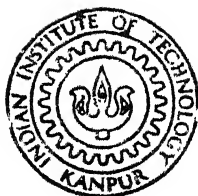
by

SAMINATHAN PONNUSAMY

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION AND CONVOLUTION TECHNIQUES TO UNIVALENT FUNCTIONS THEORY

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by

SAMINATHAN PONNUSAMY

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INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

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அம்மா, அப்பா, சிற்றப்பா திரு. சண்முகம்

அவர்களுக்கு

அன்புக் காணிக்கை

TO

MY

BELOVED PARENTS

AND

UNCLE SHRI· SHANMUGAM

CERTIFICATE

This is to certify that the work embodied in the thesis 'SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION AND CONVOLUTION TECHNIQUES TO UNIVALENT FUNCTIONS THEORY' by SAMINATHAN PONNUSAMY has been carried out under my supervision and has not been submitted elsewhere for a degree or diploma.

December, 1988.

(O.P. JUNEJA)
Professor
Department of Mathematics
Indian Institute of Technology
Kanpur, India

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CHAPTER - I

INTRODUCTION

1.1 HISTORICAL INTRODUCTION:

A function f defined in a domain D is said to be univalent in D if it is one-to-one in D, that is, f takes on no value more than once in D; in otherwords, if $f(z_1) = f(z_2)$ and $z_1, z_2 \in D$ then $z_1 = z_2$. The function $f(z) = \bar{z}$ is analytic nowhere but it is univalent in the complex plane $\mathbb{C}$. A necessary condition for an analytic function f to be univalent in D is that $f'(z) \neq 0$ for z in D which itself is not sufficient; for example, f defined by $f(z) = e^z$ is not univalent in $\mathbb{C}$ though its derivative never vanishes in $\mathbb{C}$.

Without loss of generality we assume D to be the open unit disc $U = \{z : |z| < 1\}$. Because of the Riemann mapping theorem, any simply connected domain in the complex plane $\mathbb{C}$ which is not the whole plane, can be mapped by an analytic function univalently onto the unit disc U. Thus the investigation of analytic functions which are univalent in a simply connected domain with more than one boundary point can be confined to the investigation of analytic functions which are univalent in U. This is to simplify and to give short and elegant formulae.

Let H be the class of functions f analytic in U which have the series representation around z = 0 of the form

$$(1.1.1) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Obviously, for $f \in H$, $f(0) = f'(0) - 1 = 0$. Let S denote the subclass of H consisting of univalent functions in U. Also we denote by $U_r = \{z \in \mathbb{C} : |z| < r\}$, $\partial U_r = \{z \in \mathbb{C} : |z| = r\}$ and $\bar{U}_r = U_r \cup \partial U_r = \{z \in \mathbb{C} : |z| \leq r\}$.

The Koebe function k, defined by

$$(1.1.2) \qquad k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n$$

which maps U onto the slit plane $\mathbb{C} \setminus \{(-\infty, \frac{-1}{4}]\}$ is the leading example of a function in S. It may be noted that the conformal mapping of U onto $\mathbb{C} \setminus \{te^{i\theta} : \frac{1}{4} \leq t < \infty\}$ which belongs to S is $e^{i(\theta+\pi)} k(ze^{-i(\theta+\pi)})$, a 'rotation' of k (which we will also call rotation of the Koebe function) provides solution to many extremal problems for the class S. After Koebe's [55] initiation, the early works on univalent function theory can be found in Bieberbach [12] , Gronwall [42, 43], Pick [97] and Faber [30].

In 1916, Bieberbach [13] proved that for every f in S, $|a_2| \leq 2$ with equality only for the rotation of the Koebe function (1.1.2). In a footnote he remarked that perhaps quite generally

(1.1.3) $\qquad |a_n| \leqq n, \; n = 2,3,\ldots$

holds for every f in S. This footnote became the famous Bieberbach conjecture and has been the principal stimulus for research on univalent functions.

Lowner [66], Garabedian and Schiffer [34], Pederson [95] and independently Ozawa [92], Pederson and Schiffer [96] have proved the Bieberbach conjecture for the coefficients a_n for n = 3,4,6 and 5 respectively. Besides the estimation of individual coefficients, there exists a tradition of estimates of the form $|a_n| \leqq cn + d$ for every n = 2,3,... . Such estimates (for c and d) are obtained by Littlewood [64], Milin [71], FitzGerald [31] Horowitz [46,47]. There is also a beautiful regularity theorem of Hayman [45] stating that $\lim_{n \to \infty} \dfrac{|a_n|}{n}$ exists for every f in S, and is smaller than 1 unless f is a Koebe function.

For f in S, it is often useful to look at the related function

(1.1.4) $\quad h(z) = \{f(z^2)\}^{1/2} = b_1 z + b_3 z^3 + \ldots + b_{2n-1} z^{2n-1} + \ldots, b_1 = 1,$

This is an odd function in S, and every odd function h in S can be represented as such a square root transform. In 1936, Robertson [106] conjectured that if h is an odd function in S, then

$$(1.1.5) \qquad \sum_{k=1}^{n} |b_{2k-1}|^2 \leqq n, \quad n = 2,3,\dots,(b_1 = 1)$$

and he showed that this conjecture implies the Bieberbach conjecture. Robertson's conjecture is in turn implied by the Milin conjecture [71,p.72] stating that if $f \in S$ and

$$(1.1.6) \qquad \log\left(\frac{f(z)}{z}\right) = 2 \sum_{k=1}^{\infty} c_k z^k$$

then

$$(1.1.7) \qquad \sum_{k=1}^{n} k(n+1-k)|c_k|^2 \leqq \sum_{k=1}^{n} \binom{n+1-k}{k}, \quad n = 1,2,\dots$$

Recently in 1984, Louis de Branges [26] gave a proof of the Milin conjecture (and hence of the conjectures of Bieberbach and Robertson). It is clear that de Branges miraculous proof should lead to deeper inequalities than the Milin's conjecture and is surprisingly short in view of the rich continuing history of the problem and the great effort that went into trying to settle this conjecture. It is to be noted that de Branges has shown the true depth of Lowner's work. Another simpler version of de Branges proof was given by FitzGerald and Pommerenke [32].

A detailed treatment of the information about the univalent functions are the books by Pommerenke [101], Duren [27] and two-volume book of Goodman [40]. Certain aspects of the subject have also been covered in the books by Nehari [86] , Goluzin [39] , Hayman [45], Jenkins [51] and Olli Lehto [56] . Schober's lecture notes [124] provide

an additional information about the linear space and quasi conformal connections. A recent exhaustive Bibliography of schlicht functions by Bernardi [9] containing more than 4200 references is also available in the literature on the subject till 1981.

1.2 THE CLASS P(A,B):

The various conjectures (e.g., Bieberbach conjecture, Milin conjecture, Robertson conjecture etc.) attracted eminent mathematicians to work in the theory of univalent functions. Attempts to prove or disprove these conjectures inspired researchers not only to develop elegant and useful techniques in complex analysis but also led to the introduction and study of various subclasses of univalent functions. In this section, we give some basic definitions of various subclasses of functions with positive real part.

Let B_1 be the class of functions w, analytic in U with w(0) = 0, and $|w(z)| < 1$ for z $\in$ U. The following basic result for functions in B_1 is well known.

LEMMA 1.2.1 [Schwarz Lemma] If w is in B_1, then $|w(z)| \leq |z|$, z $\in$ U equality holds if and only if w(z) = λz where $|\lambda| = 1$.

We call functions in B_1 as Schwarz functions.

Suppose f and g are analytic in U. We say that f is subordinate to g, written f $\prec$ g or f(z) $\prec$ g(z), if there exists a function w in B_1 such that f(z) = g(w(z)), z $\in$ U.

If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ [27, p.35].

We denote by $P_N(A,B)$, $-1 \leq B < 1$, $B < A$ the class of functions p of the form $p(z) = 1+p_N z^N + p_{2N}z^{2N} + \ldots$ and defined by

$$(1.2.2) \qquad p(z) = \frac{1+Aw(z)}{1+Bw(z)}$$

for some $w \in B_1$ with an expansion

$$w(z) = b_N z^N + b_{2N}z^{2N} + \ldots , \quad z \in U.$$

For convenience, we write $P_1(A,B) \equiv P(A,B)$. An application of subordination principle (see Duren [27, p.190-191]) yields that image of $|z| \leq r < 1$ under $p \in P_N(A,B)$ is contained in the disc

$$(1.2.3) \qquad |\eta - a_N| \leq d_N,$$

where

$$(1.2.4) \qquad a_N = \frac{1-ABr^{2N}}{1-B^2r^{2N}} , \quad d_N = \frac{(A-B)r^N}{1-B^2r^{2N}} .$$

It immediately follows from (1.2.4) that if $p \in P_N(A,B)$, then on $|z| = r < 1$, $(-1 \leq B < A \leq 1)$

$$(1.2.5) \qquad \frac{1-Ar^N}{1-Br^N} \leq \mathrm{Re}\{p(z)\} \leq |p(z)| \leq \frac{1+Ar^N}{1+Br^N} .$$

The inequalities are sharp for the function

$$p_0(z) = \frac{1+Az^N}{1+Bz^N} ,$$

The class $P(A,B)$ plays a significant role in the study of univalent functions. In fact, many of the problems in univalent functions may be formulated in terms of members of $P(A,B)$. The following special cases of $P(A,B)$ are of interest :

$$P(1-2\beta,-1) = \{p: \text{ Re } p(z) > \beta, \ z \in U, \ 0 \leq \beta < 1\}$$

$$P(1,-1+\frac{1}{M}) = \{p: \ |p(z)-M| < M, \ z \in U, \ M > \frac{1}{2}\}$$

$$P(1,2\beta-1) = \{p: \ |p(z) - \frac{1}{2\beta}| < \frac{1}{2\beta}, \ z \in U, \ 0 < \beta < 1\}$$

$$P(\beta,0) = \{p: \ |p(z)-1| < \beta, \ z \in U, \ 0 < \beta \leq 1\}$$

$$P(\beta,-\beta) = \{p: \ |p(z)-1|/|p(z)+1| < \beta, z \in U, \ 0 < \beta \leq 1\}.$$

Several results concerning these classes may be found in Janowski [49] McCarty [69,70], Schaffer [123]. In recent years there have been several papers in literature on $P(A,B)$ or on classes with different parametrization. With different parametrization the class $P(A,B)$ $(-1 \leq B < A \leq 1)$ was introduced and studied by Juneja and Mogra [52] also.

The function p in P $(\equiv P(1,-1))$ need not be univalent in U. Just as the Koebe function plays a crucial role in the study of class S, likewise does the function

$$(1.2.6) \qquad p(z) = \frac{1+z}{1-z} = 1+2 \sum_{n=1}^{\infty} z^n$$

in the study of the class P.

1.3 CERTAIN SUBCLASSES OF H :

We consider next the subclasses of H consisting of
starlike and convex mappings. A domain D in the complex
plane is called starlike with respect to $w_o \in D$ provided
that $tw + (1-t)w_o \in D$ when $w \in D$ and $0 \le t \le 1$. In other
words, if a point is in D, then so is the line segment
connecting that point to w_o. Let S* denote the class of
those analytic functions f in H for which f(U) is starlike
with respect to O. The family S* is analytically characterized
by $f \in H$ and

$$(1.3.1) \qquad Re\ \{\frac{zf'(z)}{f(z)}\} > 0, \quad z \in U$$

or equivalently $f \in S^*$ if and only if $\frac{zf'}{f} \in P(1,-1) \equiv P$.
From

$$(1.3.2) \quad \frac{\partial}{\partial\Theta} \{arg\ f(re^{i\Theta})\} = Re\ \{re^{i\Theta}\frac{f'(re^{i\Theta})}{f(re^{i\Theta})}\},$$

$$(z=re^{i\Theta}, 0<r<1, 0\le\Theta < 2\pi)$$

it follows that (1.3.1) expresses the local fact that the
argument of the vector $f(re^{i\Theta})$ increases with Θ.

The family S*(β) (called starlike functions of order
β < 1) can be defined by $f \in H$ and

$$Re\ \{\frac{zf'(z)}{f(z)}\} > \beta, \quad z \in U, \beta < 1.$$

For $0 \leq \beta < 1$, $S^*(\beta)$ is clearly a subclass of S^* and it is readily seen that $f \in S^*(\beta)$ $(0 \leq \beta < 1)$ if and only if $\frac{zf'}{f} \in P(1-2\beta,-1)$.

In recent years various subclasses of S^* have been considered by different researchers in the field. Prominant amongst these being the classes $S^*(A,B)$ $(-1 \leq B < A \leq 1)$, $S^*(1-2\beta,-1)$ $(0 \leq \beta < 1)$, $S^*(1,-1 + \frac{1}{M})$ $(M > \frac{1}{2})$, $S^*(\beta,0)$ $(0 < \beta \leq 1)$, $S^*(\beta,-\beta)$ $(0 < \beta \leq 1)$ for which the function $f \in H$ is such that $\frac{zf'}{f}$ is respectively in $P(A,B)$, $P(1-2\beta,-1)$ $(0 \leq \beta < 1)$, $P(1,-1 + \frac{1}{M})$ $(M > \frac{1}{2})$, $P(\beta,0)$ $(0 \leq \beta < 1)$ and $P(\beta,-\beta)$ $(0 < \beta \leq 1)$. Several results on these subclasses of starlike functions may be found in Robertson [107], Janowski [49], McCarty [70], Padmanabhan [93], Bhargava and Nanjunda Rao [10,11], Al-Amiri [3], etc.

A domain D in the complex plane is said to be convex if for every pair of points w_1, w_2 in D, the line segment joining them is also in D. Let K denote the class of all analytic functions f in H for which $f(U)$ is convex. The family K also has an analytic characterization, viz., f is in K if and only if $f \in H$ and

$$(1.3.4) \qquad \text{Re}\ \{1 + \frac{zf''(z)}{f'(z)}\} > 0,\ z \in U.$$

As a local condition, (1.3.4) is equivalent to the assertion that the tangent to the curve $w = f(re^{i\theta})$, $0 \leq \theta < 2\pi$, turns counterclockwise as θ increases. This follows because the

angle of the tangent to this curve is arg $\{izf'(z)\}$ and

(1.3.5) $\quad \dfrac{\partial}{\partial\Theta} \{arg\ (ire^{i\Theta}f'(re^{i\Theta}))\} = \dfrac{\partial}{\partial\Theta} \{Im\ log\ (ire^{i\Theta}f'(re^{i\Theta}))\}$

$$= Re\ \{1 + \dfrac{re^{i\Theta}f''(re^{i\Theta})}{f'(re^{i\Theta})}\}\ .$$

Analogously, $f \in H$ is said to be convex of order β, $\beta < 1$, denoted by $K(\beta)$, if and only if

(1.3.6) $\quad Re\ \{1 + \dfrac{zf''(z)}{f'(z)}\} > \beta,\ z \in U,$

or equivalently $\{1 + \dfrac{zf''(z)}{f'(z)}\} \in P(1-2\beta,-1)$. A comparison of (1.3.1) and (1.3.4) shows that the map $f \longrightarrow g$ defined by $g(z) = \int_{0}^{z} f(t)t^{-1}$ dt establishes a one-to-one correspondence from S^* onto K. Similarly, $f \in K(\beta)$ if and only if $zf' \in S^*(\beta)$. The classes $S^*(\beta)$ and $K(\beta)$ for $0 \leq \beta < 1$ were introduced by Robertson [107]. We say that a function f in H is in $K(A,B)$ if and only if $zf' \in S^*(A,B)$. Clearly $K(A,B)$ is a subclass of K and $K(1-2\beta,-1) \equiv K(\beta)$ $(0 \leq \beta < 1)$. As noted earlier, for different choices of A and B, we may obtain different subclasses of K.

Strohhäcker [143] and Marx [68] proved that every convex function is starlike of order 1/2 and that the result is sharp. Later, Jack [48] proposed a general problem, viz., if $f \in K(\lambda)$ $(0 \leq \lambda < 1)$, find $\beta = \beta(\lambda)$ such that $f \in S^*(\beta)$. Jack himself gave a partial answer to this problem. However, Goel [35] and MacGregor [67] solved this problem independently.

Their rusult runs as follows : If we let $f \in K(\lambda)$ $(0 \leq \lambda < 1)$, then $f \in S^*(\beta(\lambda))$ where

$$(1.3.7) \quad \beta(\lambda) \; = \; \begin{cases} \dfrac{1-2\lambda}{2^{2(1-\lambda)}\,[1-2^{2\lambda-1}]} & , \text{ if } \lambda \neq 1/2 \\[2ex] 1/2 \log 2 & , \text{ if } \lambda = 1/2 \, . \end{cases}$$

A natural generalization of starlikeness leads to the class of spirallike functions which gives another useful criterion for univalence.

A logarithmic spiral is a curve in the complex plane of the form

$$(1.3.8) \qquad w = w_0 e^{-at} , \quad -\infty < t < \infty$$

where w_0 and a are complex constants with $w_0 \neq 0$ and Re $\{a\} \neq 0$. If we take $a = e^{i\lambda}$, $\dfrac{-\pi}{2} < \lambda < \dfrac{\pi}{2}$, the curve (1.3.8) is called a λ-spiral which joins a given point $w_0 \neq 0$ to the origin.

A domain D containing the origin is said to be λ-spirallike if for each point $w_0 \neq 0$ in D the arc of the λ-spiral form w_0 to the origin lies entirely in D. A function analytic in U, with $f(0) = 0$, is said to be λ-spiral [141] if $f(U)$ is λ-spirallike. 0-spirallike functions are simply the starlike functions. Analytically, a λ-spiral function $f \in H$ is characterized by

$$(1.3.9) \quad \text{Re } \{e^{i\lambda} \frac{zf'(z)}{f(z)}\} > 0, \; z \in U, \; |\lambda| < \pi/2.$$

Libera [62] introduced the class of λ-spiral functions of order β ($0 \le \beta < 1$), denoted by $S^{\lambda}(\beta)$, by replacing the condition (1.3.9) by

(1.3.10) $\text{Re } \{e^{i\lambda} \frac{zf'(z)}{f(z)}\} > \beta \cos \lambda$, $z \in U.$

Another interesting subclass of S which contains S^* is the class of close-to-convex functions introduced by Kaplan [54].

A function f, analytic in U, is said to be close-to-convex, if there exists a convex function g not necessarily normalized, such that

(1.3.11) $\text{Re } \{\frac{f'(z)}{g'(z)}\} > 0$, $z \in U.$

We shall denote by C, the class of close-to-convex functions f normalized by $f(0) = 0 = f'(0) - 1$. Every close-to-convex function is univalent. It may be observed that $K \subset S^* \subset C \subset S.$ However, a close-to-convex function need not be in S^* as can be seen by considering the function f defined by
$f(z) = \frac{i-1}{2} \cdot \frac{z}{1-z} - \frac{(1+i)}{2} \log (1-z)$, $z \in U.$ It may also be noted that the function h defined by
$h(z) = z \exp \{(i-1) \log (1-iz)\}$, $z \in U$, where log denotes the principal branch of logarithmic function, is $\frac{\pi}{4}$ -spirallike but is not close-to-convex in U.

Taking $g(z) = z$ in (1.3.11), we see that the class C reduces to the class R of functions f in H such that

(1.3.12)   Re $\{f'(z)\} > 0$, $z \in U$.

Let $R(\beta)$ $(\beta < 1)$ denote the class of functions $f \in H$ such that Re $\{f'(z)\} > \beta$, $z \in U$. It is clear that for $0 \leq \beta < 1$, $R(\beta) \subset R$ and that $R(\beta)$ consists of univalent functions only. For further works on R we refer to [146, 65].

Yet another subclass of C, the class of functions starlike with respect to symmetric points, is due to Sakaguchi [120].

A function f in H is said to be starlike with respect to symmetric points [120] if for every r close to 1, $r < 1$ and every z_0 on $|z| = r$, the angular velocity of f(z) about the point $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction, i.e.,

(1.3.13)   Re $\{\dfrac{zf'(z)}{f(z)-f(-z_0)}\} > 0$, for $z = z_0$, $|z_0| = r$.

Analytically, it has been characterized [120] (see also [108]) that, a function f in H is univalent and starlike with respect to symmetric points if and only if,

(1.3.14)   Re $\{\dfrac{zf'(z)}{f(z)-f(-z)}\} > 0$, $z \in U$.

Motivated by (1.3.13), the class of functions starlike with respect to N-symmetric points has been introduced and studied by Rattan Chand [22], Prithvipal Singh [132] and Reddy [104].

Mocanu [83] defined the following class by means of linear combination of starlikeness and convexity that has

occupied wide attention in the recent years (see also [121]).

A function f ∈ H is said to be α-convex in U if
$f(z) \frac{f'(z)}{z} \neq 0$ and

(1.3.15)   Re $\{(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha(\frac{zf''(z)}{f'(z)} + 1)\} > 0$, z ∈ U.

The class of all such functions is denoted by M_α. This
definition is meaningful if α is real. However, for α
real, Miller et al. [79] have shown that if f is in M_α then
for α ≥ 1, f is convex while if α < 1, then f is starlike.

Bajpai and Silvia [5] introduced the concept of
order for α-convex functions.

An explicit construction for a very wide class of
functions analytic and univalent in U was given by Bazilevič
[7] (see also [100], [125],...) as follows:

Let g be a normalized starlike function in U, h be
an analytic function in U such that h(0) = 1 and
Re $\{e^{i\nu} h(z)\} > 0$ for some real ν. Then if α > 0 and β
real, the function f, defined by,

(1.3.16) $f(z) = [\int_0^z g^\alpha(t)h(t) t^{i\beta-1} dt]^{1/(\alpha+i\beta)}$

is single-valued, analytic and univalent in U and is called
Bazilevič function of type (α,β), this class being denoted
by M(α+iβ) (Powers in (1.3.16) are meant as principal values).
Thus, f is in M(α+iβ) if and only if, there exists a g ∈ S*
and a real ν such that

(1.3.17) $\text{Re} \{ e^{i\nu} \dfrac{[f(z)]^{\alpha+i\beta-1} \, f'(z)}{[g(z)]^{\alpha} \, z^{i\beta-1}} \} > 0, \; z \in U.$

The functions f that arise from (1.3.16) when $h(z) \equiv 1$, must satisfy

(1.3.18) $\text{Re} \{ 1 + \dfrac{zf''(z)}{f'(z)} + (\alpha+i\beta-1) \dfrac{zf'(z)}{f(z)} \} > 0, \; z \in U.$

Conversely, if $f \in H$, $f(z)f'(z)/z \neq 0$ in U and f satisfies
(1.3.18) for $\alpha > 0$, then the function can be written in the
form (1.3.16) with $h(z) \equiv 1$.   In [28], Eenigenburg et al.
showed that if a function $f \in H$ with $f(z)f'(z)/z \neq 0$ in U
satisfies (1.3.18) then $f \in S^{\lambda}(0)$ where λ satisfies
$\lambda = \text{arg} \, (\alpha+i\beta)$, $\dfrac{-\pi}{2} < \lambda < \dfrac{\pi}{2}$.   It may be noted that for $\alpha = \dfrac{1}{\alpha'}$
and $\beta = \dfrac{\tan \beta'}{\alpha'}$, (1.3.18) takes the form

(1.3.19) $\text{Re} \{ \alpha' \cos \beta' (1 + \dfrac{zf''(z)}{f'(z)}) + (e^{i\beta'} - \alpha' \cos \beta') \dfrac{zf'(z)}{f(z)} \} > 0,$

$z \in U$

for some $\alpha' > 0$ and $|\beta'| < \dfrac{\pi}{2}$.   For $\beta' = 0$, (1.3.19) reduces
to the condition (1.3.15) for α'-convex functions.

1.4 <u>CONVOLUTION OF ANALYTIC FUNCTIONS</u> :

Recently, there has been considerable interest in
studying the properties of Hadamard product (or convolution)
of analytic functions in U.   Given two analytic functions
$f(z) = \sum\limits_{n=1}^{\infty} a_n z^n$, $(|z| < R_1)$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n$, $(|z| < R_2)$,
their Hadamard product is defined as the power series

$$(1.4.1) \qquad (f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n$$

and represents an analytic function in $|z| < R_1 R_2$.

In 1958, Polya and Schoenberg [98] conjectured that if f and g are convex in U, then so is f * g. In 1966, Suffridge [144] proved that the convolution (f * g) of two convex functions f and g, is univalent and infact close-to-convex. In 1973, Rusheweyh and Sheil-Small [117] proved the Polya-Schoenberg conjecture in its entirety. Infact, they proved the following:

THEOREM 1.4.1 :

(i)   If f and g $\in$ K, then f * g $\in$ K.

(ii)   If f $\in$ K and g $\in$ C, then f * g $\in$ C.

(iii) If f and g $\in$ S*(1/2), then f * g $\in$ S*($\frac{1}{2}$).

They also proved the following stronger subordination conjecture of Wilf [149].

THEOREM 1.4.2 : Let φ and $\Psi \in$ K and suppose that f $\prec \Psi$. Then $\varphi * f \prec \varphi * \Psi$.

The proof of these conjectures has earned keen interest in the use of convolution techniques and the mechanics developed have been successfully applied to various problems since then.

Using the concept of convolution, Ruscheweyh [112] introduced an interesting class K_n of functions f $\in$ H satisfying

(1.4.2)   Re $\{\frac{D^{n+1}f(z)}{D^nf(z)}\} > \frac{1}{2}, z \in U, n \in N_o \equiv N \cup \{0\} \equiv \{0,1,2,\ldots\}$,

where $D^nf(z) = \frac{z}{(1-z)^{n+1}} * f(z) \equiv z \frac{d^n}{dz^n}(z^{n-1}f(z))$.

We see that $K_o \equiv S^*(\frac{1}{2})$ and $K_1 \equiv K$. Ruscheweyh [112], alongwith other results, proved the containment relation

(1.4.3) $K_{n+1} \subset K_n$, $n \in N_o$.

In [1,2], Al-Amiri investigated the class $K_n(\alpha)$ of functions $f \in H$, $f(z)f'(z) \neq 0$ in $0 < |z| < 1$, satisfying,

(1.4.4)   Re $\{J_n(f;\alpha)\} > \frac{1}{2}$, $z \in U$

for some $\alpha \geq 0$, where

(1.4.5) $J_n(f;\alpha) = (1-\alpha) \frac{D^{n+1}f(z)}{D^nf(z)} + \alpha \frac{D^{n+2}f(z)}{D^{n+1}f(z)}$, $z \in U$

and showed that for each n $\in N_o$, and for $\alpha \geq 0$

(1.4.6) $K_n(\alpha) \subset K_n(0)$

(1.4.7) $K_n(\alpha) \subset K_n(\beta)$ for $\alpha > \beta \geq 0$

(1.4.6), for $\alpha = 0$, reduces to (1.4.3).

Further results on convolution in geometric function theory can be found in Ruscheweyh [112,114,115], Sheil-Small [125,127,128], Suffridge [144], Al-Amiri [2], Lewis [60], Silverman and Silvia [129], Silverman et al.[131], and others.

1.5 DIFFERENTIAL SUBORDINATION :

In this section we give a brief account of a very recent powerful concept called differential subordination. The strength and usefulness of this notion lies in the fact that not only simple and short proofs of some of the classical results can be given by this approach but many of the earlier results also get an improved and sharpened form by this method. Further, a number of new and interesting applications are obtained to the theory of univalent functions. The work in this direction has been initiated by Miller and Mocanu [74,75] and developed by Eenigenburg et al. [28,29], Miller and Mocanu [76,77,78], Bulboacă [15,16,17,18,19,20] and others. The following lemmas are quite useful to develop the theory in this direction.

LEMMA 1.5.1 : Let g defined by $g(z) = g_n z^n + g_{n+1} z^{n+1} + \ldots$, $n \geq 1$, $z \in U$ be analytic in the unit disc U with $g_n \neq 0$, and let $z_0 \neq 0$ be a point of U such that

$$(1.5.1) \qquad |g(z_0)| = \max_{|z| \leq |z_0|} |g(z)|.$$

Then there is a real number m, $m \geq n \geq 1$, such that

(i) $z_0 \dfrac{g'(z_0)}{g(z_0)} = m$ and $\mathrm{Re}\left\{1 + z_0 \dfrac{g''(z_0)}{g'(z_0)}\right\} \geq m.$

Part (i) of the lemma is due to Jack [48] while part (ii) is due to Miller and Mocanu [74].

LEMMA 1.5.2 : Let p defined by $p(z) = a + p_n z^n + \ldots$, $n \geq 1$, be analytic in the unit disc U, and let q defined by $q(z) = a + q_1 z + \ldots$ be analytic and univalent in $\bar{U}$. If p is not subordinate to q in U, then there exist a real number m ($m \geq n \geq 1$), $z_0 \in U$ and $\zeta_0 \in \partial U$, such that

(i) $p(|z| < |z_0|) \subset q(U)$,

(ii) $p(z_0) = q(\zeta_0)$,

(iii) $\arg [z_0 p'(z_0)] = \arg [\zeta_0 q'(\zeta_0)]$,

(iv) $|z_0 p'(z_0)| = m\, |\zeta_0 q'(\zeta_0)| > 0$,

(v) $\operatorname{Re} \{1 + \dfrac{z_0 p''(z_0)}{p'(z_0)}\} \geq m \operatorname{Re} \{1 + \dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\}$.

Let Ψ be a complex valued holomorphic function defined in a domain D of $\mathbb{C}^3$ i.e. $\Psi : D \subset \mathbb{C}^3 \longrightarrow \mathbb{C}$ and let h be analytic and univalent in U. Suppose p is analytic in U, $(p(z), zp'(z), z^2 p''(z)) \in D$ when $z \in U$, and p satisfies the second order differential subordination

(1.5.2) $\Psi(p(z), zp'(z) , z^2 p''(z)) \prec h(z), \ z \in U.$

The analytic and univalent function q is said to be dominant of (1.5.2) if $p \prec q$ in U for all p satisfying (1.5.2). If $\tilde{q}$ is a dominant of (1.5.2) and $\tilde{q} \prec q$ for all dominants q of (1.5.2), then $\tilde{q}$ is said to be the best dominant of (1.5.2) (It may be noted that if there are two best dominants $\tilde{q}_1$ and $\tilde{q}_2$, then $\tilde{q}_1 \prec \tilde{q}_2$ and $\tilde{q}_2 \prec \tilde{q}_1$. This implies that

$\tilde{q}_1(z) = \tilde{q}_2(e^{i\Theta}z)$. Hence the best dominant of (1.5.2), if it exists, will be unique upto rotation.

Miller and Mocanu [76,77] investigated the properties of first order differential subordination, i.e. they replaced (1.5.2) by

$$(1.5.3) \qquad \varphi(p(z), zp'(z)) \prec h(z), \ z \in U.$$

If we take φ in (1.5.3) to be $\varphi(r,s) = r + \dfrac{s}{\beta r + \nu}$ ($r = p(z)$, $s = zp'(z)$) then (1.5.3) is called Briot-Bouquet differential subordination and this very special case has been considered by Ruscheweyh and Singh [118], Eenigenburg et al. [29], Miller and Mocanu [77] and others to obtain many interesting applications in the theory of univalent functions.

1.6   INTEGRAL TRANSFORMS :

The class preserving integral operators defined on the classes K, S^*, C, S^λ etc. and related problems have been extensively investigated in recent years. For $f \in H$, Libera [61] defined the integral transform by

$$(1.6.1) \qquad [I_{1,1}(f)](z) = [I_1(f)](z) = \frac{2}{z} \int_0^z f(t)\, dt$$

and showed that

$$(1.6.2) \qquad I_1(S^*) \subseteq S^*; \ I_1(K) \subseteq K; \ I_1(C) \subseteq C.$$

Bernardi [8] showed that the above results continue to hold for the more general transform

$$(1.6.3) \qquad F_{1,c}(z) = [I_{1,c}(f)](z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

where $c \in N = \{1,2,3,\ldots\}$. Singh [135] considered transform

$$(1.6.4) \quad F_{\beta,c}(z) = [I_{\beta,c}(f)](z) = [\frac{\beta+c}{z^c} \int_0^z t^{c-1} f^\beta(t)dt]^{1/\beta}$$

where $\beta = 1,2,\ldots,$ $c = 1,2,\ldots,$ and showed that $I_{\beta,c}[S^*] \subset S^*$. Bajpai and Srivastava [6] extended the results of Bernardi to $S^*(\lambda)$ and $K(\lambda)$ $(0 \leq \lambda < 1)$. From Lewandowski et al. [58] it follows that (1.6.2) continues to hold for $F_{1,c}$ if c in (1.6.3) is taken to be complex number satisfying Re $c \geq 0$. Ruscheweyh [111] considered the operator defined by (1.6.4) and showed that for $\beta > 0$ and a complex number c such that Re $c \geq 0$, the operator (1.6.4) satisfies $I_{\beta,c}(S^*) \subset S^*$.

In a paper , Miller et al. [80] studied the integral transforms defined by

$$(1.6.5) \quad (L.f)(z) = [\frac{\beta+c}{z^c \, \Phi(z)} \int_0^z f^\alpha(t) \, \varphi(t) \, t^{\delta-1} \, dt]^{1/\beta}$$

where $\beta+c = \alpha+\delta$. Various selections of the parameters $\alpha, \beta,$ c and δ , and the functions Φ , f , φ give some of the earlier results as special cases.

1.7 MEROMORPHIC UNIVALENT FUNCTIONS :

In earlier sections we have considered certain subclasses of analytic univalent functions in U. We would now review analogous subclasses for meromorphic univalent functions in $\{z : 0 < |z| < 1\} \equiv U \setminus \{0\}$.

Thus closely related to S is the class Σ of functions

(1.7.1)
$$g(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n$$

which are univalent and analytic in $0 < |z| < 1$ except for
a simple pole at $z = 0$.

A function g in Σ is said to be meromorphic starlike
of order β ($\beta < 1$) if the complement of g(U) is starlike of
order β with respect to the origin. We shall denote the
class of meromorphic starlike functions of order β by $\Sigma^*(\beta)$
and $\Sigma^*(0) \equiv \Sigma^*$. A function g in Σ is said to belong to $\Sigma^*(\beta)$
($\beta < 1$) if and only if

(1.7.2)
$$- \text{Re } \{\frac{zg'(z)}{g(z)}\} > \beta, \ z \in U.$$

A function g in Σ is said to be meromorphic convex of order β
($\beta < 1$) if and only if

(1.7.3)
$$- \text{Re } \{1 + \frac{zg''(z)}{g'(z)}\} > \beta, \ z \in U.$$

We denote the class of meromorphic convex functions of order β
by $\Sigma_K(\beta)$ and $\Sigma_K(0) \equiv \Sigma_K$. It is clear that if g is in $\Sigma_K(\beta)$,
then $-zg'(z)$ is in $\Sigma^*(\beta)$.

The analogue of λ-spiral function for meromorphic
functions can be defined as follows.

A function $g \in \Sigma$ is said to be λ-spiral of order β
($\beta < 1$) in U if there is a real $\lambda(|\lambda| < \frac{\pi}{2})$ such that

$(1.7.4)$ $\quad - \text{Re} \{e^{i\lambda} \frac{zg'(z)}{g(z)}\} > \beta \cos \lambda$, $z \in U.$

We say that a function Ψ of the form $(1.7.1)$ analytic in $0 < |z| < 1$, is close-to-convex in $0 < |z| < 1$ if there exists a function φ, $\varphi(z) = \frac{d_{-1}}{z} + \sum\limits_{n=0}^{\infty} d_n z^n$ satisfying $(1.7.2)$, for $\beta = 0$ such that

$(1.7.5)$ $\qquad \text{Re} \{\frac{z \, \Psi'(z)}{\varphi(z)}\} > 0$, $z \in U.$

It was proved by Clunie [25] that if g given by

$g(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} c_n z^n$ is in Σ^*, then

$(1.7.6)$ $\qquad |c_n| \leqq \frac{2}{n+1}$, $n = 1,2,...$

whereas Zamorski [153] showed that if $g(z) = \frac{1}{z} + \sum\limits_{n=1}^{\infty} c_n z^n$ is λ-spirallike of order 0, then

$$|c_n| \leqq \frac{2 \cos \lambda}{n+1} , \quad n = 1,2,... \, .$$

It may be noted that Clunie [25] assumed $c_0 = 0$, but as remarked by Libera and Robertson [63], Clunie's method can be modified to include the case $c_0 \neq 0$ for $g \in \Sigma$.

The class of meromorphic starlike functions has been extensively studied by Pommerenke [99], Kaczmarski [53], Miller [72], and others.

1.8   CONTENTS OF THE THESIS :

The univalent analytic functions have played a very important role in the development of the geometric theory of functions of a complex variable. Here we have described only those aspects of the theory in the direction of which we have pursued the study further. Thus, in the present work, an attempt has been made to have a unified and detailed study of various subclasses of univalent analytic functions by employing different techniques. In a number of cases, our approach not only yields a generalization of the various known results but also gives rise to many new and refined best estimates. The results are presented in the next six chapters.

Chapter II deals with some applications of convolution techniques and Briot-Bouquet differential subordination. To begin with, a unified approach to the study of various subclasses of starlike functions has been made by introducing and studying the wider class $T_{\delta,\alpha}(A,B)$, ($\delta \geq -1$, $\alpha \geq 0$ and $-1 \leq B < 1$ with $B < A$). After obtaining the containment relation for the classes $T_{\delta,\alpha}(A,B)$, a necessary and sufficient conditions interms of convolution, has been obtained for f to be in $T_{\delta,0}(A,B)$. That this class is closed under convolution with different subclasses of convex functions has also been established. Further, in addition to the investigation of Libera and Bernardi transforms of this class, sharp results have also been obtained for more general

integral transforms of certain other subclasses of univalent
analytic functions. For different values of the parameters
δ, α, A and B, the results of this chapter either improve or
yield, along with new results, the results of Al-Amiri [1 , 2],
Libera [61], Bulboacǎ [19], Puscheweyh [112], Singh and
Singh [137], Mocanu et al. [85] and others.

In Chapter III, we introduce new subclasses $H_{\delta, \alpha}(h)$
where $\delta \geq -1$, Re $\alpha \geq 0$ and h is a convex univalent function
in U. After showing the containment relation, certain
integral transforms of functions in these classes have been
considered. We also present an application of the Bernardi
integral operator to the generalized hypergeometric functions.
The results of this chapter generalize and improve the
corresponding works of Al-Amiri [1 , 2], Libera [61],
Bulboacǎ [19], Ozaki [91], Singh and Singh [137] and others.

Chapter IV is devoted to obtain some sufficient
conditions for convexity, starlikeness, λ-spirallikeness,
univalence etc. of some classes of analytic functions. The
concepts of dominant and best dominant of differential
subordination have been used to obtain some interesting
applications concerning Bazilevič functions. Further in
addition to finding results concerning a subclass of
Bazilevič functions, some integral transforms of functions
of this class have also been studied. The results, along with
certain new results, improve and generalize the results obtained
by Lewandowski et al.[57,58],Eenigenburg et al. [28],Singh [135],

Owa and Obradović [89], Yoshikawa and Yoshikai [151],
Libera [61] and others.

In Chapter V we study certain subclasses related to
N-symmetric points namely $S_N^*(A,B)$, $K_N(A,B)$ and $C_N(A,B)$
$(-1 \leq B < A \leq 1)$ where N is a positive integer. We find
the structural formulae that characterize functions in these
classes. It is also shown that these classes are not only
closed with respect to convolution with convex functions
but de la Valleé poussin means and partial sums of functions
in these classes also belong to the corresponding classes
under certain conditions. A neighbourhood result for the
class $S_N^*(A,B)$ $(-1 \leq B < A \leq 1)$ also finds a place in this
chapter. Results of this chapter include the results of
Silverman et al. [131], Silverman and Silvia [130],
Ruscheweyh [116], Ruscheweyh and Sheil-Small [117],
Stankiewicz [142], Reddy [104], Rajasekaran [103] and others.

Chapter VI is devoted to the study of third order
differential inequalities in the complex plane. The conditions
on the class of functions $h(r,s,t,u)$ defined in a domain D of
$\mathbb{C}^4$ such that

$$\{h(w(z),zw'(z), z^2w''(z),z^3w'''(z)) : |z| < 1\} \subset \Omega$$

$$\text{implies } |w(z)| < 1 \text{ for } |z| < 1$$

and

$$\{h(w(z),zw'(z),z^2w''(z),z^3w'''(z)) : |z| < 1\} \subset \Omega',$$

$$\text{implies } \operatorname{Re}\{w(z)\} > 0, \text{ for } |z| < 1$$

where $w(z)$ is in a certain class of analytic functions, Ω and Ω' are sets in the complex plane, have been obtained.

The results obtained in this chapter along with some new results extend the results of Miller and Mocanu [74].

The concluding chapter, that is, Chapter VII deals with certain subclasses $\Sigma_\delta^*(A,B)$, $\Sigma^*(A,B)$ and $\Sigma^*(A,B)$ $(-1 \leq B < 1, B < A)$ of meromorphic univalent functions of the form $g(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n$ $(0 < |z| < 1)$. The chapter starts by finding necessary and sufficient conditions interms of convolutions for the classes $\Sigma^*(A,B)$ and $\Sigma^K(A,B)$ $(-1 \leq B < A \leq 1)$. The effect of certain integral transforms, involving a complex parameter, over the classes $\Sigma_\delta^*(A,B)$, $\Sigma^*(A,B)$ and $\Sigma^K(A,B)$ has been studied to obtain perhaps for the first time, sharp results in this direction. A sort of inverse problem for these classes has also been considered. Along with some new results, the results found in this chapter improve the results of Bajpai [4], Goel and Sohi [37], Reddy and Juneja [105] and others.

CHAPTER - II

SOME APPLICATIONS TO BRIOT-BOUQUET DIFFERENTIAL SUBORDINATION

2.1 INTRODUCTION :

For $n \in N_o$, let K_n denote the class of functions $f \in H$ satisfying

(2.1.1) $\qquad \text{Re} \{\frac{D^{n+1}f(z)}{D^n f(z)}\} > \frac{1}{2}$, $z \in U$

where $D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z}{n!} \frac{d^n}{dz^n} (z^{n-1}f(z))$.

In [112] it was shown that

(2.1.2) $\qquad K_{n+1} \subset K_n$, $n \in N_o$

holds. Since K_o equals $S^*(1/2)$, the univalence of the members in K_n will be a consequence of (2.1.2). Furthermore, $K_1 \equiv K$ [the class of convex functions]. Therefore (2.1.2) is an extension of Strohhaker's result $K \subset S^*(1/2)$ [143].

In [1], Al-Amiri generalized the class K_n by introducing the class $K_n(\alpha)$, $n \in N_o$, $\alpha \geq 0$. Thus, a function $f \in H$ is said to belong to the class $K_n(\alpha)$, if the condition

(2.1.3) $\qquad \text{Re} \{J_n(f,\alpha)\} > \frac{1}{2}$,

holds, where $J_n(f,\alpha)$ is defined by (1.4.5). Al-Amiri [1] showed that for each $n \in N_o$, and for $\alpha \geq 0$,

(2.1.4) $\qquad K_n(\alpha) \subset K_n(0)$

(2.1.5) $\qquad K_n(\alpha) \subset K_n(\beta)$ for $\alpha > \beta \geq 0$.

The case $\alpha = 1$ of (2.1.4) yields (2.1.2).

In [36], Goel and Sohi tried to generalize the classes K_n and $K_n(\alpha)$ by introducing respectively the classes $T_{n,\beta}$ and $T_{n,\beta}(\alpha)$, $0 \leq \beta \leq \frac{1}{2}$, $\alpha \geq 0$, $n \in N_0$. Thus, a function $f \in H$ is said to belong to the class $T_{n,\beta}(\alpha)$, if for $0 \leq \beta \leq \frac{1}{2}$, $\alpha \geq 0$, the condition

(2.1.6) $\qquad$ Re $\{J_n(f,\alpha)\} > \beta$

holds, $J_n(f,\alpha)$ being given by (1.4.5). Here $T_{n,\beta}(0) \equiv T_{n,\beta}$. Goel and Sohi claimed [36, Theorem 1 and Theorem 5] that

(2.1.7) $\quad T_{n,\beta}(\alpha) \subset T_{n,\beta}$ for all $n \in N_0, \alpha \geq 0$ and $0 \leq \beta \leq \frac{1}{2}$.

Unfortunately, the above inclusion (2.1.7) is not valid even for particular choices of n, α and β as seen below.

For instance, if one chooses $n = 0$, $\alpha = 1$ and $\beta = \frac{1}{4}$ then (2.1.7) yields

(2.1.8) $\quad$ Re $\{1 + \frac{zf''(z)}{f'(z)}\} > -\frac{1}{2}$ implies Re $\{\frac{zf'(z)}{f(z)}\} > \frac{1}{4}$ for $z \in U$.

However, for $f(z) = \frac{1}{2}[1 - (1+z)^{-2}]$, it is easily seen that

Re $\{1 + \frac{zf''(z)}{f'(z)}\} > -\frac{1}{2}$, $z \in U$ whereas

(2.1.9) $\qquad \frac{zf'(z)}{f(z)} = \frac{1+z}{1+\frac{z}{2}} - \frac{2z}{1+z}$.

If $z \neq -1$ and $|z| = 1$ then Re $\{\frac{2z}{1+z}\} = 1$. Thus, as $w = \frac{1+z}{1 + \frac{1}{2}z}$

vanishes at $z = -1$, there are numbers z sufficiently close

to -1 so that $|z| = 1$ and Re $\{\frac{zf'(z)}{f(z)}\}$ is as close to -1 as

we like. Accordingly, there are points in U so that

Re $\{\frac{zf'(z)}{f(z)}\} < 0$ and, therefore, f is not even starlike

(although univalent) in U. This contradicts (2.1.8).

Similarly if one chooses $n = 0$, $\alpha = 1$ and $\beta = 0$

then (2.1.7) gives

(2.1.10) Re $\{\frac{zf''(z)}{f'(z)} + 1\} > -1$ implies Re $\{\frac{zf'(z)}{f(z)}\} > 0$, for $z \in U$

However (2.1.10) is not true can be seen by the same example

as considered before.

We now introduce, by using convolution techniques,

a class as follows. This class generalizes all the above

mentioned classes.

DEFINITION 2.1.1 : Let A,B,α and δ be arbitrary fixed real

numbers such that $-1 \leq B < 1$, $B < A$, $\alpha \geq 0$ and $\delta \geq -1$. A

function $f \in H$ is said to be in the class $T_{\delta,\alpha}(A,B)$ if it

satisfies

(2.1.11) $J_\delta(f,\alpha) \prec \frac{1+Az}{1+Bz}$, $z \in U$

where

(2.1.12) $J_\delta(f,\alpha) = (1-\alpha) \frac{D^{\delta+1}f(z)}{D^\delta f(z)} + \alpha \frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}$,

and $D^\delta f(z) = (z/(1-z)^{\delta+1}) * f(z)$.

It is to be noted that the function $h(z) = \frac{1+Az}{1+Bz}$ is convex in U for $A,B \in \mathbb{C}$ with $A \neq B$ and $|B| \leq 1$.

The condition (2.1.11) is equivalent to saying that $J_\delta(f,\alpha) \in P(A,B)$. It is readily seen that $T_{0,0}(A,B) \equiv S^*(A,B)$, $T_{0,1}(\frac{A+B}{2},B) \equiv K(A,B)$. Further, it is clear that $T_{n,0}(0,-1)$, $n \in N_0$ is the class K_n defined by (2.1.1) [112], whereas the class $T_{n,0}(\frac{1-n}{1+n},-1)$ is the class of functions f in H that satisfy the condition

$$(2.1.13) \qquad \text{Re } \{\frac{D^{n+1}f(z)}{D^n f(z)}\} > \frac{n}{n+1} , z \in U.$$

The class $T_{n,0}(\frac{1-n}{1+n},-1)$ ($\equiv R_n$ in the notation of [137]) has been introduced and studied by Singh and Singh [137]. The class $T_{n,\alpha}(1-2\beta,-1)$ ($0 \leq \beta \leq 1/2$) is the class considered by Goel and Sohi [36]. The classes $T_{n,\alpha}(0,-1)$ and $T_{\delta,0}(0,-1)$ were considered by Al-Amiri [1 , 2]. Further taking $\delta = 0$, $\alpha = 2\mu/(1+\mu)$ ($\mu \geq 0$), $A = 1-2\mu/(1+\mu)$, $B = -1$, it is seen that the class $T_{\delta,\alpha}(A,B)$ reduces to the class of μ-convex functions M_μ defined by (1.3.15).

Recently many of the classical results in univalent function theory have been improved and sharpened by the powerful technique of Briot-Bouquet differential subordination (see e.g. [29], [77], [85], etc.). Recall that a function p analytic in U with a power series of the form $p(z)=1+p_1z+...$ is said to satisfy Briot-Bouquet differential subordination if

$$(2.1.14) \qquad p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z), \quad z \in U$$

for β and ν complex constants and h a convex univalent function with $h(0) = 1$ and Re $\{\beta h(z) + \nu\} > 0$ in U.

The univalent function q is said to be a dominant of the Briot-Bouquet differential subordination (2.1.14) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1.14). If $\tilde{q}$ is a dominant of (2.1.14) and $\tilde{q}(z) \prec q(z)$ for all other dominants q of (2.1.14), then $\tilde{q}$ is said to be the best dominant.

In this chapter, we propose to give some applications of Briot-Bouquet differential subordination which would not only improve and sharpen many of the earlier results for the classes $S^*(A,B)$, $K(A,B)$, $K_n(\alpha) \equiv T_{n,\alpha}(0,-1)$, $T_{n,\alpha}(1-2\beta,-1)$ etc., but would also give rise to a number of new results for the other classes as well. This is accomplished by studying the wide class $T_{\delta,\alpha}(A,B)$ introduced above. In Section 2.2, we first show the sharp containment relation for $f \in T_{\delta,\alpha}(A,B)$. In Section 2.3, using convolution techniques, we obtain a necessary and sufficient condition for a function $f \in H$ to be in $T_{\delta,0}(A,B)$. In Section 2.4, it has been shown that the class $T_{\delta,0}(A,B)$ is closed under certain integral transforms. Furthermore, an application of Briot-Bouquet differential subordination to the investigation of Libera and Bernardi transforms of this class leads, perhaps for the first time, to sharp results in this direction. Further use of Briot-Bouquet differential subordination to the investigation of Libra and Bernardi transforms of this class leads, perhaps for the first time,

to sharp results in this direction. Further use of Briot-Borquet differential subordination has been made to prove sharp results for a more general integral transform for the classes $S^*(A,B)$ and $K(A,B)$ respectively. Finally, using the differential subordination, we improve and generalize results of Singh and Singh [138], and Mocanu [84].

2.2 CONTAINMENT RELATION :

In order to prove the main theorems of this section, we need the following lemmas

LEMMA 2.2.1 [17, Corollary 3.2] If A,B, β are reals with $\beta \neq 0$, $-1 \leq B < 1$ and $A \neq B$, and complex number ν satisfies

$$\text{Re } \nu \geq -\frac{\beta(1-A)}{2} \quad \text{when } B = -1 \text{ and } A \neq -1 \quad \text{and}$$

$$\text{Re } \nu \geq \max \{-\frac{\beta(1-A)}{1-B}, -\beta\frac{(1+A)}{1+B}\} \text{ whenever } |B| < 1 \text{ and } A \neq B,$$

then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \nu} = \frac{1+Az}{1+Bz}$$

has a univalent solution given by

$$(2.2.1) \quad q(z) = \begin{cases} \dfrac{z^{\beta+\nu}(1+Bz)^{\beta((A-B)/B)}}{\beta \displaystyle\int_0^z t^{\beta+\nu-1}(1+Bt)^{\beta((A-B)/B)}dt} - \dfrac{\nu}{\beta} & \text{if } B \neq 0 \\[4em] \dfrac{z^{\beta+\nu}\exp(\beta Az)}{\beta \displaystyle\int_0^z t^{\beta+\nu-1}\exp(\beta At)dt} - \dfrac{\nu}{\beta} & \text{if } B = 0. \end{cases}$$

If p(z) is analytic in U and satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \frac{1+Az}{1+Bz} \quad , \ z \in U$$

then $p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}$ and q(z) is the best dominant of the above differential subordination.

LEMMA 2.2.2 : Let $\mu(t)$ be a positive measure on the unit interval $I = [0,1]$. Let $g(t,z)$ be a complex valued function defined on $[0,1] \times U$, and integrable in t for each $z \in U$ ~~$g(t,z)$ be analytic in U for each $t \in [0,1]$ with~~ and for almost all $t \in [0,1]$, and suppose that Re $\{g(t,z)\} > 0$ on U and $g(z) = \int_I g(t,z) d\mu(t)$. If, for fixed $(0 \leq \lambda < 2\pi)$, $g(t,re^{i\lambda})$ is real for r real and

$$\text{Re } \{\frac{1}{g(t,z)}\} \geq \frac{1}{g(t,re^{i\lambda})} \ , \ \text{for } |z| \leq r \text{ and } t \in [0,1]$$

then Re $\{\frac{1}{g(z)}\} \geq \frac{1}{g(re^{i\lambda})}$ for $|z| \leq r$ and $0 \leq \lambda < 2\pi$.

Proof : The method of proof of this lemma is based on the lines of proof of Wilken and Feng [150]. It is immediate that $g(z)$ is analytic, Re $\{g(z)\} > 0$ in U, and $g(re^{i\lambda})$ is real. Further, it is easy to see that the following two statements are equivalent

$$\text{Re } \{\frac{1}{g(z)}\} \geq \frac{1}{g(re^{i\lambda})} \ , \ \text{for } |z| \leq r, \ (0 \leq \lambda < 2\pi)$$

$$|g(z) - \frac{1}{2} g(re^{i\lambda})| < \frac{1}{2} g(re^{i\lambda}), \ \text{for } |z| \leq r.$$

Thus $|g(z) - \frac{1}{2}g(re^{i\lambda})| = |\int_I g(t,z)d\mu(t) - \frac{1}{2}\int_I g(t,re^{i\lambda})d\mu(t)|$

$$\leqq \int_I |g(t,z) - \frac{1}{2}g(t,re^{i\lambda})|d\mu(t)$$

$$\leqq \int_I \frac{1}{2}g(t,re^{i\lambda})d\mu(t) = \frac{1}{2} g(re^{i\lambda}).$$

So, by the above equivalent statements we have

$$\text{Re } \{\frac{1}{g(z)}\} \geqq \frac{1}{g(re^{i\lambda})} \ , \ |z| \leqq r.$$

This completes the proof of the lemma.

For a,b,c real numbers other than $0,-1,-2,\ldots$ the hypergeometric series

$$(2.2.2) \quad F(a,b;c;z) = 1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2\ c(c+1)} z^2 + \ldots$$

represents an analytic function in U [148, p.281]. The following identities are well known.

LEMMA 2.2.3 [148, Chapter XIV] : For a,b,c real numbers other than $0,-1,-2,\ldots$ and $c > b > 0$ we have

$$(2.2.3) \quad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\ \Gamma(c-b)}{\Gamma(c)} F(a,b;c;z)$$

$$(2.2.4) \quad F(a,b;c;z) = F(b,a;c;z)$$

$$(2.2.5) \quad F(a,b;c;z) = (1-z)^{-a}F(a,c-b;c;z/(1-z))$$

$$(2.2.6) \quad F(a,b;(\frac{a+b+1}{2});\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\ \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\ \Gamma(\frac{b+1}{2})} \ .$$

We now state and prove

THEOREM 2.2.1 : Let $-1 \leq B < 1$, $B < A$, $\delta > -1$ and $0 < \alpha < \delta+2$ satisfy

$$(2.2.7) \qquad A' \equiv [(\delta+2)A - B\alpha]/(\delta+2-\alpha) \leq 1$$

i.e., $\qquad \dfrac{\delta+2}{\alpha}(1-A) - (1-B) \geq 0.$

(a) Then

$$(2.2.8) \qquad T_{\delta,\alpha}(A,B) \subset T_{\delta,0}(A',B).$$

Further, for $f \in T_{\delta,\alpha}(A,B)$ we also have

$$(2.2.9) \qquad \frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \frac{\alpha}{\delta+2-\alpha}\left[\frac{1}{Q(z)}\right] \equiv \tilde{q}(z), \ z \in U$$

where

$$(2.2.10) \quad Q(z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{\frac{\delta+2}{\alpha}\left(\frac{A-B}{B}\right)} t^{\frac{\delta+2}{\alpha}-2}\, dt & \text{if } B \neq 0, \\[4ex] \displaystyle\int_0^1 \exp\left(\frac{\delta+2}{\alpha}(t-1)Az\right) t^{\frac{\delta+2}{\alpha}-2}\, dt & \text{if } B = 0. \end{cases}$$

(b) If in addition to (2.2.7) one has $-1 \leq B < A < 0$, then

$$(2.2.11) \qquad T_{\delta,\alpha}(A,B) \subset T_{\delta,0}(1-2\rho_1, -1)$$

where $\qquad \rho_1 = \left[F\left(1, \dfrac{\delta+2}{\alpha}\left(\dfrac{B-A}{B}\right); \dfrac{\delta+2}{\alpha}; \dfrac{-B}{1-B}\right)\right]^{-1}.$

The result is sharp.

(c) If in addition to (2.2.7) one has $0 < B < 1$, $B < A < 2B$, then

(2.2.12) $\qquad T_{\delta,\alpha}(A,B) \subset T_{\delta,0}(1-2\rho_2,-1)$

where $\qquad \rho_2 = [F(1, \frac{\delta+2}{\alpha} (\frac{A-B}{B}); \frac{\delta+2}{\alpha} ; \frac{B}{1+B})]^{-1}.$

The result is sharp.

Proof : We follow the method similar to that of Mocanu et al. [85]. Since, for $\delta > -1$,

(2.2.13) $\qquad D^\delta f(z) = z + \sum_{n=2}^{\infty} \frac{\overline{|(n+\delta)}}{(n-1)! \ \overline{|(1+\delta)}} a_n z^n,$

it can be easily verified that

(2.2.14) $\qquad z(D^\delta f(z))' = (\delta+1) D^{\delta+1}f(z) - \delta D^\delta f(z).$

Let $f \in T_{\delta,0}(A,B)$ where $\delta > -1$, $\alpha > 0$, $-1 \leq B < 1$ and $B < A$.

Set $\quad g(z) = z [D^\delta f(z)/z]^{1/(\delta+1)}$ and

$r_1 = \sup \{r : g(z) \neq 0, 0 < |z| < r < 1\}$.

Using (2.2.14) it follows that

(2.2.15) $\qquad p(z) = \frac{zg'(z)}{g(z)} = \frac{D^{\delta+1} f(z)}{D^\delta f(z)}$

is analytic in $|z| < r_1$ and $p(0) = 1$. Since $f \in T_{\delta,\alpha}(A,B)$, (2.1.11) coupled with (2.2.14) easily leads to

(2.2.16) $\qquad P(z) + \frac{zP'(z)}{\beta P(z)+\nu} \prec \frac{1+Az}{1+Bz}$, $|z| < r_1$

where

(2.1.17) $P(z) = (1 - \frac{1}{\beta})p(z) + \frac{1}{\beta}$, with $\beta = \frac{\delta+2}{\alpha}$ and $\nu = -1.$

Using Lemma 2.2.1 we deduce that

$$(2.2.18) \qquad P(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \ , \quad |z| < r_1$$

where $q(z)$ is the best dominant of (2.2.16) and is given by (2.2.1). Again by (2.2.17) we get

$$(2.2.19) \quad p(z) \prec \frac{\alpha}{\delta+2-\alpha} \left[\frac{1}{Q(z)}\right] \equiv \tilde{q}(z) \prec \frac{1 + \frac{1}{\beta-1}(A\beta-B)z}{1+Bz}$$

where $Q(z)$ is given by (2.2.10). By (2.2.17) and (2.2.18), we see from (2.2.15) that $g(z)$ is starlike (univalent) in $|z| < r_1$. Thus it is not possible that $g(z)$ vanishes in $|z| < r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore $p(z)$ is analytic in U. However (2.2.19) implies $p(z) \prec \tilde{q}(z)$ in U. Hence by (2.2.15) $f \in T_{\delta,\alpha}(A,B)$ implies $\frac{D^{\delta+1}f(z)}{D^\delta f(z)} \prec \tilde{q}(z)$ provided δ, α, A and B satisfy (2.2.7). This proves (2.2.8) and (2.2.9).

(b) Suppose in addition to (2.2.7) one has $-1 \leqq B < A < 0$, then we show that

$$(2.2.20) \qquad \underset{|z|<1}{\text{Inf}} \ \{\text{Re } \tilde{q}(z)\} = \tilde{q}(-1).$$

If we set $a = \beta\left(\frac{B-A}{B}\right)$, $b = \beta+\nu$ $(\beta = \frac{\delta+2}{\alpha}$, $\nu = -1)$, $c = b+1$ then $c > b > 0$. From (2.2.10) by using (2.2.3), (2.2.4) and (2.2.5) we see that for $B \neq 0$

$$Q(z) = (1+Bz)^a \int_0^1 (1+Btz)^{-a} \, t^{b-1} \, dt$$

$$= (1+Bz)^a \frac{\overline{\lceil(b)} \ \overline{\lceil(c-b)}}{\overline{\lceil(c)}} (1+Bz)^{-a} F(a,c-b;c;Bz/(1+Bz))$$

$$(2.2.21)$$

$$= \frac{\overline{\lceil(b)} \ \overline{\lceil(c-b)}}{\overline{\lceil(c)}} F(a,c-b;c;Bz/(1+Bz))$$

To prove (2.2.20) we show that Re $\{\frac{1}{Q(z)}\} \geqq \frac{1}{Q(-1)}$, z $\in$ U.
Again (2.2.10), by (2.2.21) for $-1 \leqq B < A < 0$ (so that
c > a > 0), can be rewritten as

$$Q(z) = \int_0^1 g(t,z) \ d\mu(t)$$

where

$$(2.2.22) \qquad g(t,z) = \frac{1+Bz}{1+(1-t)Bz} \quad \text{and}$$

$$(2.2.23) \qquad d\mu(t) = \frac{\overline{\lceil(b)}}{\overline{\lceil(a)} \ \overline{\lceil(c-a)}} t^{a-1}(1-t)^{c-a-1} \ dt.$$

For $-1 \leqq B < A < 0$, it may be noted that Re $\{g(t,z)\} > 0$,
g(t,-r) is real for $0 \leqq r < 1$, t $\in [0,1]$ and

$$\text{Re} \ \{\frac{1}{g(t,z)}\} = \text{Re} \ \{\frac{1+(1-t)Bz}{1+Bz}\} \geqq \frac{1-(1-t)Br}{1-Br} = \frac{1}{g(t,-r)}$$

for $|z| \leqq r < 1$ and t $\in [0,1]$. Therefore, using Lemma 2.2.2,
we deduce that Re $\{\frac{1}{Q(z)}\} \geqq \frac{1}{Q(-r)}$, $|z| \leqq r < 1$ and by letting
r $\to 1^-$ we obtain Re $\{\frac{1}{Q(z)}\} \geqq \frac{1}{Q(-1)}$, z $\in$ U. This proves
(2.2.20) which, in view of (2.2.19), leads to (2.2.11).
Since $\tilde{q}(z)$ being the best dominant of (2.2.16) with p(z)
satisfying (2.2.17) and $\tilde{q}(z)$ satisfying (2.2.20), the
sharpness of the result now follows.

(c) To prove (2.2.12) we have to show that

(2.2.24)   Inf Re $\{q(z)\} = \tilde{q}(1)$

provided the real numbers δ, α, A, B with $0 < B < 1$ and $B < A < 2B$
satisfy (2.2.7) also for $\delta > -1$ and $\alpha > 0$.

If we set $a = \beta(\frac{A-B}{B})$ with $0 < B < 1$ and $B < A < 2B$;
$b = \beta + \nu$, $c = \beta + \nu + 1$ ($\beta = \frac{\delta+2}{\alpha}$, $\nu = -1$) then $c > b > 0$ as
well as $c > a > 0$. As in part (b)

$$Q(z) = \frac{\overline{|}(b) \; \overline{|}(c-b)}{\overline{|}(c)} \; F(1,a;c;Bz/(1+Bz))$$

$$= \int_0^1 g(t,z) \, d\mu(t)$$

where $g(t,z)$ and $d\mu(t)$ are respectively given by (2.2.22)
and (2.2.23) with $0 < B < 1$ and $B < A < 2B$.

For $0 < B < 1$, it may be noted that Re $\{g(t,z)\} > 0$
in U, $g(t,r)$ is real for $0 \leq r < 1$, $t \in [0,1]$ and

$$\text{Re } \{\frac{1}{g(t,z)}\} \geq \frac{1+(1-t)Br}{1+Br} = \frac{1}{g(t,r)}$$

for $|z| \leq r < 1$ and $t \in [0,1]$. Therefore, using Lemma 2.2.2,
we deduce that Re $\{\frac{1}{Q(z)}\} \geq \frac{1}{Q(r)}$, $|z| \leq r < 1$ and by letting
$r \to 1^-$ we obtain Re $\{\frac{1}{Q(z)}\} \geq \frac{1}{Q(1)}$, $z \in U$. This, by (2.2.19),
leads to (2.2.24). This shows the containment relation
(2.2.12) holds true and the result is sharp. This completes
the proof of the theorem.

COROLLARY 2.2.1 : For $f \in H$, $\delta > -1$ and max $(-1, 1-(\frac{\delta+2}{\alpha})) \leq B < 0$, we have the sharp result

$$(1-\alpha) \frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} + \alpha \frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} \prec \frac{1}{1+Bz} , \quad z \in U$$

implies

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \frac{1}{1+Bz} , \quad z \in U.$$

Proof : Taking A = 0 in the above theorem, from (2.2.9), we obtain for max $\{-1, -\frac{(\delta+2)}{\alpha}\} \leq B < 0$ that

$$(2.2.24) \quad f \in T_{\delta,\alpha}(0,B) \text{ implies } \frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \tilde{q}_1(z), \quad z \in U$$

where $\tilde{q}_1(z) = \frac{\alpha}{\delta+2-\alpha} [(1+Bz)^{\frac{\delta+2}{\alpha}} \int_0^1 t^{\frac{\delta+2}{\alpha}-2} (1+Btz)^{-(\frac{\delta+2}{\alpha})} dt]^{-1}.$

Now using the identity, which follows from (2.2.3] for a = c, namely

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-c} dt = \frac{\Gamma(b) \Gamma(c-b)}{\Gamma(c)} (1-z)^{-b}$$

we obtain

$$\tilde{q}_1(z) = \frac{\alpha}{\delta+2-\alpha} [(1+Bz)^{\frac{\delta+2}{\alpha}} \{ \frac{\Gamma(\frac{\delta+2}{\alpha}-1) \Gamma(1)}{\Gamma(\frac{\delta+2}{\alpha})}(1+Bz)^{-(\frac{\delta+2}{\alpha}-1)} \}]^{-1}$$

i.e. $\tilde{q}_1(z) = \frac{1}{1+Bz}$.

Now the required implication follows from (2.2.24).

Taking $\alpha = 1$ and $B = -1$ in the above corollary it follows that for $f \in H$ and $\delta \geq 0$,

(2.2.25) $\quad$ Re $\dfrac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} > \dfrac{1}{2}$, $z \in U$ implies Re $\dfrac{D^{\delta+1}f(z)}{D^{\delta}f(z)} > \dfrac{1}{2}$, $z \in U$.

The case $\delta = n \in N_0$ of (2.2.25) we obtained by Ruscheweyh [112, Theorem 1]. Taking $A = 1-2\rho$ and $B = -1$, in Theorem 2.2.1 (for the case $A = 0$ see Corollary 2.2.1) the following corollary not only gives the correct form of the containment relation (2.1.7) but also shows that it is not possible to improve it further.

COROLLARY 2.2.2 : $\quad$ Let $\delta > -1$ and $0 < \dfrac{\alpha}{\delta+2} \leqq \rho < 1$.

(a) $\quad$ Then

$$T_{\delta,\alpha}(1-2\rho,-1) \subset T_{\delta,0}\left(1-2\left(\dfrac{(\delta+2)\rho-\alpha}{\delta+2-\alpha}\right), -1\right).$$

Further, if $f \in T_{\delta,\alpha}(1-2\rho,-1)$, then

$$\dfrac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \prec \dfrac{\alpha}{\delta+2-\alpha}\left[\int_0^1 \left(\dfrac{1-tz}{1-z}\right)^{-2\left(\frac{\delta+2}{\alpha}\right)(1-\rho)} t^{\frac{\delta+2}{\alpha}-2} dt\right]^{-1}.$$

(b) $\quad$ Furthermore if $\max\left\{\dfrac{\alpha}{\delta+2}, \dfrac{1}{2}\right\} \leqq \rho < 1$ then

$$T_{\delta,\alpha}(1-2\rho,-1) \subset T_{\delta,0}(1-2\rho_2,-1)$$

where $\quad \rho_2 = \left[F\left(1,2\left(\dfrac{\delta+2}{\alpha}\right)(1-\rho); \dfrac{\delta+2}{\alpha}; \dfrac{1}{2}\right)\right]^{-1}$.

The result is sharp.

$\qquad$ Taking $\delta = 0$, $\alpha = 2\mu/(\mu+1)$ and $\rho = \mu/(\mu+1)$ in Corollary 2.2.2, by (2.2.6), we find that if f is μ convex ($\mu \geqq 1$), then $f \in S^*\left(\Gamma(\frac{2+\mu}{2\mu})/\sqrt{\pi}\ \Gamma(\frac{\mu+1}{\mu})\right)$, a result due to Miller et al. [81]. Similarly if we take $\delta = 0$, $\alpha = 1$ and $\rho = \dfrac{1+\lambda}{2}$ we obtain from Corollary 2.2.2 that for $0 \leqq \lambda < 1$, $f \in K(\lambda)$ implies $f \in S^*(\beta(\lambda))$ where $\beta(\lambda)$, given by (1.3.7), can be obtained by expanding $F(1,2(1-\lambda);2;1/2)$. This is due to Goel [35], MacGregor [67] and Zmorovich et al. [154].

The above Corollary 2.2.2 for $\ell = n \in N_0$, is due to Bulboacă [19].

The classes $T_{\delta,\alpha}(A,B)$ have been defined for $\delta \geq -1$. However, in Theorem 2.2.1, δ has been taken to satisfy $\delta > -1$. The following theorem shows that it is possible to obtain an extension of Theorem 2.2.1 for the case $\delta = -1$ and α real with $\alpha > 0$.

THEOREM 2.2.2 : Let f $\in$ H, $\delta \geq -1$, h be a convex univalent function in U with h(0) = 1. Then for a complex number α satisfying Re $\alpha > 0$,

$$(1-\alpha)\ \frac{D^{\delta+1}f(z)}{z} + \alpha\ \frac{D^{\delta+2}f(z)}{z} \prec h(z), \quad z \in U$$

implies

$$\frac{D^{\delta+1}f(z)}{z} \prec (\tfrac{\delta+2}{\alpha})z^{-(\frac{\delta+2}{\alpha})} \int_0^z h(t)\ t^{(\frac{\delta+2}{\alpha})-1}\ dt \prec h(z), \quad z \in U.$$

For the proof of the above theorem, we need the following well-known result due to Hallenbeck and Ruscheweyh [44] (see also [75]).

LEMMA 2.2.4 : Let p defined by p(z) = 1 + $p_1 z$ + ... be analytic in U and h be a convex univalent function in U with h(0) = 1 and ν be a complex number such that Re $\nu > 0$, then

$$p(z) + \frac{zp'(z)}{\nu} \prec h(z), \quad z \in U$$

implies

$$p(z) \prec q(z) \equiv \nu z^{-\nu} \int_0^z h(t)\ t^{\nu-1}\ dt \prec h(z), \quad z \in U$$

and q(z) is the best dominant.

<u>Proof of Theorem 2.2.2</u> :   Set $p(z) = \frac{D^{\delta+1}f(z)}{z}$.   Then p is obviously analytic in U and p(0) = 1. Using the identity (2.2.14), a little manipulation leads to

$$(1-\alpha)\,\frac{D^{\delta+1}f(z)}{z} + \alpha\,\frac{D^{\delta+2}f(z)}{z} = p(z) + (\frac{\alpha}{\delta+2})\,zp'(z).$$

Now the theorem follows from hypothesis and Lemma 2.2.4.

By giving different values to the parameters δ , α and choosing suitable convex univalent function h in the above theorem we get the improved form of the results obtained by Chichra [24], Singh and Singh [140] and others. For instance, $\delta = -1$.

<u>COROLLARY 2.2.3</u>:

(i)   <u>For</u> $f \in H$, Re $\alpha \geq 0$ $(\alpha \neq 0)$ <u>we have</u>

$$(1-\alpha)\,\frac{f(z)}{z} + \alpha f'(z) \prec 1+\lambda z, \quad (\lambda \neq 0)$$

<u>implies</u> $\frac{f(z)}{z} \prec 1 + (\frac{\lambda}{\alpha+1})z$, $z \in U$

(ii)   For $f \in H$ and $A \neq B$ $(|B| \leq 1)$

$$(1-\alpha)\,\frac{f(z)}{z} + \alpha f'(z) \prec \frac{1+Az}{1+Bz}$$

<u>implies</u> $\frac{f(z)}{z} \prec (\frac{1}{\alpha})z^{-(1/\alpha)} \int_0^z (\frac{1+At}{1+Bt})\,t^{\frac{1}{\alpha}-1}\,dt \prec \frac{1+Az}{1+Bz}, z \in U.$

In the above corollary if we choose $\alpha = 1$ we obtain

$f \in H$ and $f'(z) \prec \frac{1+Az}{1+Bz}$ implies $\frac{f(z)}{z} \prec \begin{cases} 1 + \frac{A}{2}z \text{ if } B = 0, \\[2mm] \frac{A}{B} + (\frac{B-A}{B})\frac{1}{Bz}\log(1+Bz) \text{ if } B \neq 0, \end{cases}$

$z \in U.$

Further for B = -1 and A = 1-2β, (β < 1) we obtain the sharp result f ∈ H and Re {f'(z)} > β implies Re {$\frac{f(z)}{z}$} >β+(1-β)(2 ln2-1) where a use has been made of the fact that the function Q defined by Q(z) = -1 - $\frac{2}{z}$ log(1-z), (z ∈ U), is univalent and maps U onto a convex domain and Re {Q(z)} > 2 ln 2-1, z ∈ U (see Robertson [110]). From the above relation, we easily obtain the sharp result

f ∈ H and Re {f'(z)} > -($\frac{2 \ln 2-1}{2(1-\ln 2)}$) implies Re {$\frac{f(z)}{z}$} > 0, z ∈ U

which improves the well known result.

Taking δ = n ∈ N_o, α = 1 and h(z) = $\frac{1+z}{1-z}$ in the above theorem we obtain that for f ∈ H,

Re {$\frac{D^{n+2}f(z)}{z}$} > 0 implies $\frac{D^{n+1}f(z)}{z}$ $\prec$ $\frac{(n+2)}{z^{n+2}}$ $\int_0^z (\frac{1+t}{1-t})t^{n+1}dt$ $\prec$ $\frac{1+z}{1-z}$,

$$z \in U.$$

This improves the Theorem 1 of Singh and Singh [140].

THEOREM 2.2.3 : <u>Let</u> f ∈ H <u>and</u> δ ≥ -1. <u>Then</u>

(2.2.26) $\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)}$ - 1 $\prec$ h(z), z ∈ U

<u>implies</u> [$\frac{D^{\delta+1}f(z)}{z}$]$^{1/(\delta+2)}$ $\prec$ q(z), z ∈ U

<u>provided</u> h <u>given by</u> h(z) = a_1z + ... (a_1 ≠ 0) <u>be starlike in</u> U <u>and that</u> q(z) = exp {$\int_0^z t^{-1}h(t)dt$} <u>is univalent.</u>
<u>The result is sharp.</u>

<u>Proof</u> : We define the analytic function p by

$$(2.2.27) \qquad p(z) = [\frac{D^{\delta+1}f(z)}{z}]^{1/(\delta+2)} .$$

Clearly $p(0) = 1$. Logarithmic derivation of (2.2.27) gives by using (2.2.14) and (2.2.26)

$$(2.2.28) \qquad \frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - 1 = \frac{zp'(z)}{p(z)} \prec h(z)$$

where $h \in S^*$. The differential equation

$$(2.2.29) \qquad \frac{zq'(z)}{q(z)} = h(z)$$

has analytic solution given by $q(z) = \exp\{\int_0^z t^{-1} h(t)dt\}$ and this is univalent by hypothesis. Now to prove the theorem it is sufficient to show that $p(z) \prec q(z)$ in U. Suppose that p is not subordinate to q in U; then, by Lemma 1.5.2, there is a point $z_0 \in U$ and $\zeta_0 \in \partial U$, and $m \geq 1$ such that $p(z_0) = q(\zeta_0)$ and $zp'(z_0) = m\zeta_0 q'(\zeta_0)$. Thus

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{m_0 \zeta_0 q'(\zeta_0)}{q(\zeta_0)} = mh(\zeta_0).$$

Since $h(U)$ is a starlike domain and $m \geq 1$ we obtain $\frac{z_0 p'(z_0)}{p(z_0)} \notin h(U)$ which is a contradiction to (2.2.28). Hence $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.2.28), i.e., the subordination relation is sharp. Hence the theorem.

REMARK 2.2.1 : (i) For $f \in H$, $h(z) = \frac{z}{1-z}$ and $\delta = n \in N_0 \cup \{-1\}$, the above theorem yields that

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec \frac{1}{1-z}, \quad z \in U \text{ implies } [\frac{D^{n+1}f(z)}{z}]^{1/(n+1} \prec \frac{1}{1-z} ,$$

a result due to Singh and Singh [140, Theorem 7].

 (ii) For $f \in H$, $h(z) = 2(1-\beta) \frac{z}{1-z}$ and $\delta = -1$, Theorem 2.2.3 gives

$$\text{Re} \left\{\frac{zf'(z)}{f(z)}\right\} > \beta \text{ implies } \frac{f(z)}{z} \prec \frac{1}{(1-z)^{2(1-\beta)}} \quad (0 \leqq \beta < 1)$$

i.e., $\frac{zf''(z)}{f'(z)} \prec 2(1-\beta)\frac{z}{1-z}$ implies $f'(z) \prec \frac{1}{(1-z)^{2(1-\beta)}}(0 \leqq \beta < 1)$.

 (iii) Taking $\delta = -1$ and then replacing f by zf' in Theorem 2.2.3, we obtain

$$\frac{zf''(z)}{f'(z)} \prec h(z) \text{ implies } f'(z) \prec \exp \left\{ \int_0^z t^{-1} h(t)dt \right\}$$

provided the conditions of Theorem 2.2.3 is satisfied. In particular, choosing $h(z) = 4ke^{i\Theta}z/(1-z^2)$ and $h(z) = z/(1+z)$ we respectively obtain,

$$(2.2.30) \quad \frac{zf''(z)}{f'(z)} \prec \frac{4k\, e^{i\Theta}z}{1-z^2} \text{ implies } f'(z) \prec (\frac{1+z}{1-z})^{2ke^{i\Theta}},$$

$$(0 < k \leqq \cos \Theta \, , \, |\Theta| \leqq \pi/2);$$

$$(2.2.31) \quad \text{Re} \left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2} \text{ implies } |f'(z)-1| < 1.$$

It may be noted that the case $\Theta = 0$ and $k = \frac{1}{2}$ of (2.2.30) gives a sufficient condition for f to be close-to-convex (univalent) in U. This condition was obtained by Miller and Mocanu [75]. On the other hand (2.2.31) which is an improvement of a result of Ozaki [91], gives a sufficient condition for a function f to be close-to-convex and bounded in U.

(iv) For f $\in$ H and δ = 0, Theorem 2.2.3 gives

$$1 + \frac{zf''(z)}{f'(z)} \prec 1+2h(z), z \in U \text{ implies } \sqrt{f'(z)} \prec \exp\{\int_0^z t^{-1}h(t)dt\},$$

$$z \in U$$

where h be starlike in U. The result is sharp.

For instance, if one chooses $h(z) = \frac{(1-\rho)z}{1-z}$ and
$h(z) = \frac{\lambda z}{2}$ ($|\lambda| \leqq 2$) one obtains respectively the sharp results,

f $\in$ H and Re $(1+\frac{zf''(z)}{f'(z)}) > \rho$ implies Re $\{\sqrt{f'(z)}\} > \frac{1}{2^{1-\rho}}$, z $\in$ U
and

f $\in$ H and $1 + \frac{zf''(z)}{f'(z)} \prec 1+\lambda z$ implies $\sqrt{f'(z)} \prec \exp\{\frac{\lambda}{2}z\}$

(The function $h(z) = e^{\lambda z}$ is convex (univalent) in U if $|\lambda| \leqq 1$
and Re $\{e^{\lambda z}\} > 0$, z $\in$ U if $|\lambda| \leqq \pi/2$.

2.3 <u>CONVOLUTION CONDITIONS</u> :

In this section, using convolution techniques we give
necessary and sufficient condition for a function f $\in$ H to be
in $T_{\delta,0}(A,B)$.

<u>THEOREM 2.3.1</u> : <u>A</u> <u>function</u> f $\in$ H <u>is</u> <u>in</u> $T_{\delta,0}(A,B)$ ($\delta > -1$,
$-1 \leqq B < A(\delta+1) - B\delta \leqq 1$) <u>if</u> <u>and</u> <u>only</u> <u>if</u>, <u>for</u> $0 < |z| < 1$,
$|x| = 1$ and $x \neq 1$

(2.3.1) $f(z) * [\frac{z+\{(1+Ax)/(B-A)x\}z^2}{(1-z)^{2+\delta}}] \neq 0.$

<u>Proof</u>: Suppose f $\in$ $T_{\delta,0}(A,B)$ for $\delta > -1$ and $-1\leqq B<A(\delta+1)-B\delta \leqq 1$.
Then

$$\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} \neq \frac{1+Ax}{1+Bx} \quad , \quad |x| = 1 \text{ and } x \neq 1$$

in $0 < |z| < 1$ which is equivalent to

$$(1+Bx)\, D^{\delta+1}f(z) - (1+Ax)\, D^{\delta}\,f(z) \neq 0.$$

Since $D^{\delta}f(z) = -\dfrac{z}{(1-z)^{\delta+1}} * f(z)$, the above equation can be rewritten as

$$f(z) * [(1+Bx)\frac{z}{(1-z)^{\delta+2}} - (1+Ax)\frac{z}{(1-z)^{\delta+1}}] \neq 0$$

which reduces to

$$(2.3.2) \quad f(z) * [\frac{(B-A)xz + (1+Ax)z^2}{(1-z)^{\delta+2}}] \neq 0.$$

Now (2.3.1) follows from (2.3.2) in case $|x| = 1$ and $x \neq 1$.

The converse part follows easily since all the steps can be retraced back.

Putting $\delta = 0$ or $\delta = 1$ in the above theorem, we obtain the recent results of Silverman and Silvia [130, Theorem 7, Corollary] for a function $f \in H$ to be respectively in $S^*(A,B)$ and $K(A,B)$.

Putting $\delta = 1-2\alpha$ $(\alpha < 1)$, $A = \frac{\alpha-\beta}{1-\alpha}$ $(0 \leq \beta < 1)$, $B = -1$ and replacing x by $-\frac{1}{x}$ in the above theorem we obtain a result of Shiel-Small et al. [128, Theorem 3].

COROLLARY 2.3.1 : A function $f \in H$ is in $T_{1-2\alpha,0}(\frac{\alpha-\beta}{1-\alpha},-1)$, $(\alpha < 1)$ if and only if

$$f(z) * \left[\dfrac{z + \{\dfrac{(1-\alpha)x + \beta-\alpha}{1-\beta}\} z^2}{(1-z)^{3-2\alpha}}\right] \neq 0, \quad |x| = 1, \quad 0 < |z| < 1.$$

The case $\alpha = \beta$ of Corollary 2.3.1 is due to Ruscheweyh [114].
Next we prove

THEOREM 2.3.2 :   If $\varphi \in K$ and $f \in T_{\delta,0}(A,B)$ then

$\varphi * f \in T_{\delta,0}(A,B)$ for $\delta > -1$ and $-1 \leq B < (1+\delta)A - B\delta \leq 1$.

For the proof of the theorem we need the following version of a Lemma due to Ruscheweyh and Sheil Small [117].

LEMMA 2.3.1 :   If $\varphi \in K$ and $g \in S^*$ then for each function F analytic in U, the image of U under $(\varphi * Fg)/(\varphi * g)$ is a subset of the convex hull of $F(U)$.

Proof of Theorem 2.3.2 :   Since $\delta > -1$, $-1 \leq B < (1+\delta)A - B\delta \leq 1$, and $f \in T_{\delta,0}(A,B)$, one has

$$(2.3.3) \qquad \frac{D^{\delta+1}f(z)}{D^\delta f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in U,$$

we see from (2.2.14) that

$$z\frac{(D^\delta f(z))'}{D^\delta f(z)} \prec \frac{1+(A(\delta+1)-B\delta)z}{1+Bz}, \quad z \in U.$$

This implies that $D^\delta f(z) = z + \sum\limits_{n=2}^{\infty} \dfrac{\Gamma(n+\delta)}{(n-1)!\,\Gamma(\delta+1)} a_n z^n$ is

a subclass of S^*.   If we set $F(z) = \dfrac{D^{\delta+1}f(z)}{D^\delta f(z)}$ and $g(z) = D^\delta f(z)$

in the above lemma, we have for a convex function φ,

$$\frac{\varphi(z) * (\dfrac{D^{\delta+1}f(z)}{D^\delta f(z)} \cdot D^\delta f(z))}{\varphi(z) * D^\delta f(z)} = \frac{D^{\delta+1}(\varphi * f)(z)}{D^\delta(\varphi * f)(z)}.$$

By Lemma 2.3.1, the range of $\dfrac{D^{\delta+1}(\varphi * f)}{D^{\delta}(\varphi * f)}$ lies in the closed convex hull of $(\dfrac{D^{\delta+1}f}{D^{\delta}f})(U)$. The theorem, now, follows from (2.3.3).

For $A = 1-2\lambda$, $B = -1$ and $\delta = 0$, the above theorem reduces to a theorem of Ruscheweyh and Sheil-Small [117]. Taking the convex function φ to be $\varphi(z) = \sum\limits_{n=1}^{\infty} \dfrac{1+c}{n+c} z^n$ or $\varphi(z) = \dfrac{1}{1-x} \log [\dfrac{1-xz}{1-z}]$, $|x| \leq 1$, $x \neq 1$, we obtain

COROLLARY 2.3.2 : If f, defined by $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, belongs to $T_{\delta,0}(A,B)$ then the function F or G defined by

$$F(z) = \dfrac{1+c}{z^c} \int_0^z t^{c-1} f(t)dt$$

$$G(z) = \int_0^z \dfrac{f(t) - f(xt)}{t-tx} dt.$$

are also belong to $T_{\delta,0}(A,B)$.

2.4 INTEGRAL TRANSFORMS :

For a function $f \in H$, Libera [61] defined the integral transform $I_{1,1}(f)$ by

(2.4.1) $F_{1,1}(z) = [I_{1,1}(f)](z) = \dfrac{2}{z} \int_0^z f(t)dt$

and showed that

(2.4.2) $f \in S^*$ or K implies $F_{1,1} \in S^*$ or K

respectively. Bernardi [8] showed that the above result (2.4.2) continues to hold for the more general integral transfo:

$$(2.4.3) \qquad F_{1,c}(z) = [I_{1,c}(f)](z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)dt$$

where $c \in N_o = \{0,1,2,3,\ldots\}$. Bajpai and Srivastava [6] extended the result of Bernardi to $S^*(\rho)$ and $K(\rho)$ $(0 \leq \rho < 1)$. From Lewandowski et al. [58] it follows that (2.4.2) continues to hold for $F_{1,c}(z)$ if c in (2.4.3) is taken to be a complex number satisfying $\mathrm{Re}(c) \geq 0$. Ruscheweyh [111] considered a more general operator than $F_{1,c}(z)$ and showed that for $\beta > 0$ and a complex number c such that $\mathrm{Re}\ c \geq 0$, the integral transform

$$(2.4.4) \quad F_{\beta,c}(z) = [I_{\beta,c}(f)](z) = [\frac{\beta+c}{z^c} \int_0^z t^{c-1} f^\beta(t)dt]^{1/\beta}$$

satisfies $I_{\beta,c}[S^*] \subset S^*$.

Ruscheweyh [112] also considered the Bernardi integral transform of function in K_n defined by (2.4.3) and showed that $I_{1,c}[K_n] \subset K_n$ provided $\mathrm{Re}\ c > \frac{n-1}{2}$. Goel and Sohi [36] attempted to extend this result for $T_{n,0}(1-2\rho,-1)$ for $\mathrm{Re}\ c \geq (1-\rho)n-\rho$ $(n \in N_o, \ 0 \leq \rho \leq \frac{1}{2})$. Al-Amiri [2] showed that $I_{1,c}[T_{\delta,0}(0,-1)] \subset T_{\delta,0}(0,-1)$ provided $\mathrm{Re}\ c > \frac{\delta-1}{2}$ whereas Singh and Singh [137] showed that

$$I_{1,1}[T_{n,0}(\frac{2-n}{1+n},-1)] \subset T_{n,0}(\frac{1-n}{1+n},-1).$$

In our next theorem by showing that the class $T_{\delta,0}(A,B)$ is preserved under the Bernardi Transform (2.4.3) we not only get refinements of aforesaid results but also show that it is not possible to improve them further.

THEOREM 2.4.1 : Let $\delta > -1$, $-1 \leq B < 1$ with $B < A$ and c be a complex number satisfying

$$(2.4.5) \qquad \text{Re } c \geqq \frac{\delta(A-B)+A-1}{1-B} \, .$$

(a) If $f \in T_{\delta,0}(A,B)$ then the function $F_{1,c}$ defined by (2.4.3) satisfies $F_{1,c} \in T_{\delta,0}(A,B)$. Furthermore we have

$$(2.4.6) \qquad \frac{D^{\delta+1} F_{1,c}(z)}{D^{\delta} F_{1,c}(z)} \prec \frac{1}{\delta+1} \left[\frac{1}{Q(z)} - (c-\delta) \right] \equiv \tilde{q}(z), \quad z \in U$$

where

$$(2.4.7) \quad Q(z) = \begin{cases} \displaystyle\int_0^1 \left\{ \frac{1+Btz}{1+Bz} \right\}^{(\delta+1)\left(\frac{A-B}{B}\right)} t^c \, dt & \text{if } B \neq 0 \\[2em] \displaystyle\int_0^1 t^c \exp \{(1+\delta)A(t-1)z\} \, dt & \text{if } B = 0. \end{cases}$$

(b) If in addition to (2.4.5) for $-1 \leq B < 0$ with $B < A$, c is real and $c > \delta-1 - (1+\delta)(A/B)$, then for $f \in T_{\delta,0}(A,B)$ we have

$$F_{1,c} \in T_{\delta,0}(1-2\rho_4,-1)$$

where

$$(2.4.8) \quad \rho_4(\delta,c,A,B) = \frac{1}{\delta+1}\left[\frac{c+1}{F(1,(1+\delta)(\frac{B-A}{B});c+2; \frac{-B}{1-B})} - (c-\delta) \right].$$

The bound is sharp.

(c) If in addition to (2.4.5), c is real and $c > (\delta+1)\frac{A}{B} - (\delta+3$ with $0 < B < 1$ and $B < A$, then for $f \in T_{\delta,0}(A,B)$ we have

$F_{1,c} \in T_{\delta,0}(1-2\rho_5,-1)$, <u>where</u>

$$(2.4.9) \quad \rho_5(\delta,c,A,B) = \frac{1}{\delta+1} \left[\frac{c+1}{F(1,(1+\delta)(\frac{A-B}{B}),c+2,\frac{B}{1+B})} - (c-\delta) \right]$$

<u>The</u> <u>result</u> <u>is</u> <u>sharp.</u>

<u>Proof</u> : Since

$$(2.4.10) \quad F_{1,c}(z) = \left[\sum_{j=1}^{\infty} \frac{1-c}{c+j} z^j \right] * f(z) \equiv F(z), \text{ say}$$

and

$$(2.4.11) \quad D^\delta f(z) = \frac{z}{(1-z)^{\delta+1}} * f(z) = \left(\sum_{n=1}^{\infty} \frac{\overline{\left|(n+\delta)\right.}}{(n-1)! \left.\overline{\left|(\delta+1)\right.}\right.} z^n \right) * f(z$$

it can be easily seen from (2.4.10) and (2.4.11) that

$$(2.4.12) \quad z(D^\delta F(z))' = (c+1)D^\delta f(z] - cD^\delta F(z).$$

We put

$$(2.4.13) \quad g(z) = z \left[\frac{D^\delta F(z)}{z} \right]^{1/(\delta+1)}$$

and $r_1 = \sup \{r : g(z) \neq 0, 0 < |z| < r\}$.   Then g is singlevalued and analytic in $|z| < r_1$ and

$$(2.4.14) \quad p(z) = \frac{zg'(z)}{g(z)} = \frac{D^{\delta+1}f(z)}{D^\delta f(z)}$$

is analytic in $|z| < r_1$, $p(0) = 1$, (2.2.14) and (2.4.12) easily lead to

$$(2.4.15) \quad (1+\delta) \frac{D^{\delta+1}f(z)}{D^\delta F(z)} + c-\delta - (1+c) \frac{D^{\delta+1}f(z)}{D^\delta F(z)} .$$

If $f \in T_{\delta,0}(A,B)$, then it is clear that $D^\delta f(z) \neq 0$ in $0 < |z| < 1$. So (2.4.14) and (2.4.15) give

$$(2.4.16) \qquad \frac{D^\delta F(z)}{D^\delta f(z)} = \frac{1+c}{c-\delta+(1+\delta)p(z)}.$$

Differentiating (2.4.14) and using (2.4.12) and (2.4.16), we get

$$(2.4.17) \qquad \frac{D^{\delta+1} f(z)}{D^\delta f(z)} = p(z) + \frac{zp'(z)}{c-\delta+(1+\delta)p(z)}, \quad |z| < r_1.$$

Since $f \in T_{\delta,0}(A,B)$, we have by (2.4.17) that

$$(2.4.18) \qquad p(z) + \frac{zp'(z)}{\beta p(z)+\nu}, \quad \beta = \delta+1, \quad \nu = c-\delta \quad .$$

Using Lemma 2.2.1, we deduce that

$$(2.4.19) \qquad p(z) \prec \tilde{q}(z) = \frac{1}{\beta Q(z)} - \frac{\nu}{\beta} \prec \frac{1+Az}{1+Bz}, \quad |z| < r_1$$

where $Q(z)$ is given by (2.4.7) and $\tilde{q}(z)$ is the best dominant of (2.4.18).

Now the function P defined by

$$P(z) = \beta\left(\frac{1+Az}{1+Bz}\right) + \nu = \frac{\beta + \nu + (\beta A + \nu B)z}{1+Bz} \quad (\beta = \delta+1, \nu = c-\delta)$$

is convex (univalent) in U and by (2.4.5) we see that Re $P(z) > 0$ for $z \in U$. This from (2.4.19) and (2.4.14) shows that it is not possible that $g(z)$ vanishes on $|z| = r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore p is analytic in U and hence by (2.4.14) and (2.4.18) we obtain the first part of the theorem.

Proceeding as in Theorem 2.2.1, the second and third parts follow.

Putting $A = 1-2\rho$, $B = -1$ in Theorem 2.4.1, we obtain

COROLLARY 2.4.1 :   Let δ and ρ be real numbers satisfying $\delta > -1$ and $\rho < 1$.

(a) If c is a complex number satisfying $\mathrm{Re}\ c \geq \delta - (1+\delta)\rho$ then $I_{1,c} \equiv F_{1,c}$ defined by (2.4.3) satisfies

$$I_{1,c}\ [T_{\delta,0}(1-2\rho,-1)] \subset T_{\delta,0}(1-2\rho,-1)$$

Furthermore for $f \in T_{\delta,0}(1-2\rho,-1)$, we have

$$\frac{D^{\delta+1}F_{1,c}(z)}{D^{\delta}\ F_{1,c}(z)} \prec \frac{1}{\delta+1}\ [\frac{1}{Q(z)} - (c-\delta)] \equiv \tilde{q}_1(z),\ z \in U$$

where $Q(z)$ is obtained from (2.4.7) with $A = 1-2\rho$ and $B = -1$

(b) If c is a real number satisfying

$$c \geq \max\ [\delta - (1+\delta)\ ,\ 2\{\delta - (1+\delta)\rho\}]$$

then we have

$$I_{1,c}\ [T_{\delta,0}(1-2\rho,-1)] \subset T_{\delta,0}(1-2\rho_6,-1)$$

where ρ_6 obtained from ρ_4 is given by

$$\rho_6 = \rho_4(\delta,c,1-2\rho,-1)$$

$$\equiv \frac{1}{\delta+1}\ [\frac{c+1}{F(1,2(1+\delta)(1-\rho);\ c+2,1/2)} - (c-\delta)].$$

The result is sharp.

The part (b) of Corollary 2.4.1 for $\delta = n \in N_0$ is due to Bulboacă[19]

REMARKS 2.4.1(i) : Taking $\rho = \frac{2n-1}{2(n+1)}$, $\delta = n \in N_0$ and $c = 1$ in part (b) of Corollary 2.4.1 we obtain a result of Singh and Singh [137, Theorem 1] which satisfies that for $f \in H$

$$\text{Re } \{\frac{D^{n+1}f(z)}{D^n f(z)}\} > \frac{2n-1}{2(n+1)} \text{ implies Re } \{\frac{D^{n+1}F_{1,1}(z)}{D^n F_{1,1}(z)}\} > \frac{n}{n+1} , z \in U$$

where $F_{1,1}$ is the Libera integral operator defined by (2.4.1) and the result is sharp. This for $n = 0$ and $n = 1$ respectively extends the results of Libera [61], namely if $f \in H$ is in $S^*(-\frac{1}{2})$ then $F_{1,1}$ defined by (2.4.1) belongs to S^* and the result is sharp for the function f defined by $f(z) = z/(1+z)^3$. Further if $f \in H$ is in $K(-\frac{1}{2})$ then $F_{1,1}$ belongs to K and the result is sharp for the function f given by $f(z) = \frac{1}{2}[1-(1+z)^{-2}]$.

(ii) It can be easily seen that by giving different values to the parameters δ , A, B and c, the results obtained in this direction by Al-Amiri [1,2], Ruscheweyh and Singh [119], Goel and Sohi [36] etc. get improved considerably in many cases.

The following theorem shows that it is possible to obtain an extension Theorem 2.4.1 to the case $\delta = -1$ also. Since this can be easily proved using Lemma 2.2.4, we omit its proof. So we state

THEOREM 2.4.2 : Let $\delta > -1$, Re $(1+c) > 0$, h be a convex univalent function in U with $h(0) = 1$ and $f \in H$. Then we have

$$\frac{D^{\delta+1}f(z)}{z} \prec h(z) \quad \underline{implies} \quad \frac{D^{\delta+1}F_{1,c}(z)}{z} \prec q(z) = \frac{1+c}{z^{1+c}} \int_0^z t^c h(t)dt$$

where $F_{1,c}$ is defined by (2.4.3). The result is sharp.

For the case $\delta = 0$ and $c = 1$, the above theorem not only generalizes an earlier result of Libera [61] but also shows that the result obtained is sharp. Further, Theorem 2.4.4 extends the result of Singh and Singh [140, Theorem 4] for suitably chosen h namely $h(z) = \frac{1+(1-2\beta)z}{1-z}, (\beta < 1)$.

In the case $f \in T_{\delta,0}(A,B)$ and c in Bernardi transform (2.4.3) of f is taken to be δ, Theorem 2.4.1 takes the following form.

THEOREM 2.4.3 : Let $\delta > -1$, $-1 \leq B < 1$ and $B < A$. If $f \in T_{\delta,0}(A,B)$ then the function $F_{1,\delta}$ defined by

$$F_{1,\delta}(z) = [I_{1,\delta}(f)](z) = \frac{1+\delta}{z^\delta} \int_0^z t^{\delta-1} f(t) \, dt$$

belongs to $T_{\delta,1}(\frac{B+A(\delta+1)}{\delta+2}, B)$.

Proof : From the definition of $F_{1,\delta}$, we have

$$(2.4.20) \quad z(D^\delta F_{1,\delta}(z))' = (1+\delta)D^\delta f(z) - \delta D^\delta F_{1,\delta}(z)$$

and by using (2.2.14) we have

$$(2.4.21) \quad z(D^\delta F_{1,\delta}(z))' = (1+\delta)D^\delta F_{1,\delta}(z) - D^\delta F_{1,\delta}(z).$$

Equating (2.4.20) and (2.4.21) we obtain

$$(2.4.22) \quad D^\delta f(z) = D^{\delta+1}F_{1,\delta}(z).$$

Also using (2.4.20) and (2.4.21), we get

(2.4.23) $D^{\delta+1}f(z) = \frac{1}{\delta+1} [(\delta+2)D^{\delta+2}F_{1,\delta}(z)-D^{\delta+1}F_{1,\delta}(z)]$.

Thus from (2.4.22) and (2.4.23) we have

(2.4.24) $(\frac{\delta+2}{\delta+1}) \dfrac{D^{\delta+2}F_{1,\delta}(z)}{D^{\delta+1}F_{1,\delta}(z)} - \dfrac{1}{\delta+1} = \dfrac{D^{\delta+1}f(z)}{D^{\delta} f(z)}$.

Since $f \in T_{\delta,0}(A,B)$, the theorem follows easily from (2.4.24). Hence the theorem.

It may be noted that for $A = 1 - 2(\frac{n}{n+1})$, $B = -1$ and $\delta = n \in N_0$, the above theorem leads to the result of Singh and Singh [137, Theorem 2].

Let k be a complex number with $Re(k) > 0$. Consider the function Q_k defined by

(2.4.25) $Q_k(z) = 2C \dfrac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2 - (z+b)^2}$, $z \in U$

where

(2.4.26) $C = \frac{1}{Re(k)} [|k| \sqrt{1+2\ Re\ k} - Im(k)]$

and b, with $|b| < 1$, is defined by $k = 2Cb/(1-b^2)$.

We note that the function Q_k can be defined by $Q_k(z) = R[(z+b)/(1+\bar{b}z)]$, where $R(z) = 2Cz/(1-z^2)$ and $b = R^{-1}(k)$. Hence Q_k is univalent in U, $Q_k(0) = k$ and $Q_k(U) = R(U)$ is the complex plane slit along the half-lines $Re\ w = 0$, $|Im\ w| \geq C$. Next we prove the following new

theorem, by weakening the condition on f for the Bernardi
transform (2.4.3) of f with c = δ in Theorem 2.4.1.

THEOREM 2.4.4 : Let H_δ be the unique function that maps U
onto the complex plane slit along the half-lines Re w = 0,
$|I_m w| \geq C_1 = \frac{\sqrt{3+2\delta}}{\delta+1}$ ($H_\delta(0) = 1$). If $\delta > -1$ and f $\in$ H
satisfies the subordination relation

(2.4.27) $\qquad \dfrac{D^{\delta+1} f(z)}{D^\delta f(z)} \prec H_\delta(z),$ in U

then $\qquad$ Re $\{\dfrac{D^{\delta+1} F_{1,\delta}(z)}{D^\delta F_{1,\delta}(z)}\} > 0$ in U

where $F_{1,\delta}$ is given by

$$F_{1,\delta}(z) = \frac{\delta+1}{z^\delta} \int_0^z t^{\delta-1} f(t) \, dt.$$

For the proof of the theorem we need the following lemma,
more general form of which may be found in [75].

LEMMA 2.4.1 : Let Ω be a set in the complex plane $\mathbb{C}$ and
let a be a complex number with Re(a) > 0. Suppose that
the function $\Psi : \mathbb{C}^2 \times U \longrightarrow \mathbb{C}$ satisfies the condition

(2.4.28) $\qquad \Psi(ir_2, s_1; z) \notin \Omega,$ for all real r_2 ,

$s_1 \leq - \dfrac{k}{2 \, Re(a)} |a-ir_2|^2$ and all z $\in$ U.

$\qquad$ If p is analytic in U, with $p(z) = a + p_k z^k + \ldots (k \geq 1)$

and

$$\Psi(p(z),zp'(z);z) \in \Omega , \quad z \in U,$$

then $\quad$ Re $p(z) > 0$ in U.

Proof of Theorem 2.4.4 : $\quad$ As in the proof of Theorem 2.4.1

if we set

$$p(z) = \frac{D^{\delta+1}F_{1,\delta}(z)}{D^{\delta} F_{1,\delta}(z)}$$

then we obtain that (2.4.27) is equivalent to

$$\Psi(p(z),zp'(z)) = p(z) + \frac{zp'(z)}{(\delta+1)p(z)} < H_\delta(z), \quad z \in U,$$

[i.e. $\quad$ The case $c = \delta$ of (2.4.17)]

where p is analytic in U, $p(z) = 1 + p_1 z + \dots$,

$\Psi(r,s) = r + \frac{s}{(1+\delta)r}$ and H_δ is as defined in the statement

of Theorem 2.4.4.

$\qquad$ We shall show that Ψ satisfies the condition (2.4.28)

of Lemma 2.4.1, i.e.,

(2.4.29) $\qquad i\left(r_2 - \frac{s_1}{(\delta+1)r_2}\right) \notin \Omega \equiv H_\delta(U)$

for all real r_2 and $s_1 \leqq - \frac{(1+r_2^2)}{2}$.

If $r_2 > 0$, then we have

$$r_2 - \frac{s_1}{(\delta+1)r_2} \geqq r_2 + \frac{1}{2(\delta+1)} \cdot \left(r_2 + \frac{1}{r_2}\right).$$

It is easy to show that the minimum value of the right hand

side of the above inequality is C_1 where C_1 is given by

$C_1 \equiv \frac{\sqrt{3+2\delta}}{\delta+1}$; similarly one can deduce that if $r_2 < 0$, then

$$r_2 - \frac{s_1}{(\delta+1)r_2} \leqq -C_1$$

and $r_2 - \frac{s_1}{(\delta+1)r_2} \longrightarrow \pm \infty$ according as $r_2 \longrightarrow 0^+$ or 0^-.

Therefore if f satisfies (2.4.27), (2.4.29) automatically holds and the conclusion of the Theorem 2.4.4 follows from Lemma 2.4.1.

Taking $\delta = 1$ in Theorem 2.4.4 we obtain

COROLLARY 2.4.2 : If H_1 is the unique function that maps U onto a complex plane slit along the half lines Re w = 0, $|\text{Im } w| \geq \frac{\sqrt{5}}{2}$ and $f \in H$ satisfies

$$2 + \frac{zf''(z)}{f'(z)} \prec 2H_1(z), \quad z \in U$$

then the Libera transform $F_{1,1}$ given by (2.4.1) satisfies
$$\text{Re } \{2 + \frac{z\, F_{1,1}''(z)}{F_{1,1}'(z)}\} > 0, \quad \text{in U.}$$

Next we give the following theorem for the more general integral transform (2.4.4).

THEOREM 2.4.5 : Let $\beta > 0$, $\beta + \nu > 0$ and consider the integral operator $I_{\beta,\nu}$ defined by

$$(2.4.30) \quad F_{\beta,\nu}(z) \equiv [I_{\beta,\nu}(f)](z) = [\frac{\beta+\nu}{z^\nu} \int_0^z t^{\nu-1} f^\beta(t)dt]^{1/\beta}.$$

Suppose that the reals $A, B \neq 0$
satisfy

$$-1 \leqq B < 1 \quad \text{and}$$

$$-B < A \leqq 1 + \frac{\nu}{\beta}(1-B)$$

then the order of starlikeness of the class $T_{\beta,\nu}[S^*(A,B)]$ is given by

$$(2.4.31) \qquad \delta(A,B;\beta,\nu) = \inf_{|z|<1} \text{Re } q(z)$$

where $q(z)$ is given by (2.2.1). Morevoer if $-1 \leq B < O$, $B < A \leqq \min \{1 + \frac{\nu}{\beta}(1-B), -(\nu+1)\frac{B}{\beta}\}$ and $f \in S^*(A,B)$ then we have

$$(2.4.32) \quad \text{Re } \{\frac{zF'_{\beta,\nu}(z)}{F_{\beta,\nu}(z)}\} \geqq q(-r) = \frac{1}{\beta}[\frac{\beta+\nu}{F(1,\beta(\frac{B-A}{B});\beta+\nu+1;\frac{-Br}{1-Br})} - \nu$$

for $|z| \leqq r < 1$ and

$$(2.4.33) \quad \delta(A,B;\beta,\nu) = q(-1) = \frac{1}{\beta}[\frac{\beta+\nu}{F(1,\beta(\frac{B-A}{B});\beta+\nu+1; \frac{-B}{1-B})} - \nu] .$$

Furthermore if $O < B < 1$, $B < A \leqq \min \{1 + \frac{\nu}{\beta}(1-B)$, $(2\beta+\nu+1)\frac{B}{\beta}\}$ and $f \in S^*(A,B)$ then we have

$$(2.4.34) \quad \text{Re } \{\frac{zF'_{\beta,\nu}(z)}{F_{\beta,\nu}(z)}\} \geqq q(r) = \frac{1}{\beta}[\frac{\beta+\nu}{F(1,\beta(\frac{A-B}{B});\beta+\nu+1;\frac{Br}{Br+1})} - \nu]$$

for $|z| \leqq r < 1$ and

$$\delta(A,B;\beta,\nu) = q(-1) = \frac{1}{\beta}[\frac{\beta+\nu}{F(1,\beta(\frac{A-B}{B});\beta+\nu+1; \frac{B}{B+1})} - \nu]$$

where q is given by (2.2.1) and $F(a,b;c;z)$ is the hypergeometric function defined by (2.2.2). The result is sharp.

Proof :   If we let p be defined by

$$p(z) = \frac{zF'_{\beta,\nu}(z)}{F_{\beta,\nu}(z)} = \frac{z\{I_{\beta,\nu}(f)\}'(z)}{[I_{\beta,\nu}(f)](z)} \quad,\text{ then from (2.4.30)}$$

we obtain

$$p(z) + \frac{zp'(z)}{\beta p(z)+\nu} = \frac{zf'(z)}{f(z)} \quad.$$

Since $f \in S^*(A,B)$ is equivalent to $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Az}{1+Bz}$, we deduce

that p satisfies the differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \frac{1+Az}{1+Bz} \quad,\ z \in U$$

and hence by Lemma 2.2.1 $p(z) \prec q(z)$ and $q(z)$ is given by

(2.2.1). This in turn proves (2.4.31). For the proof of

(2.4.32), (2.4.33) and (2.4.34) we can proceed the same

method as that of Theorem 2.2.1. So we omit the details

of the proof. Hence the theorem.

REMARK 2.4.2 :   An analogous· problem can be considered for

convexity also. Consider the integral operator

$$(2.4.35)\quad [I_{1,\nu}(f)](z) = F_{1,\nu}(z) = \frac{1+\nu}{z^{\nu}} \int_{o}^{z} t^{\nu-1} f(t)dt,\ \nu > -1.$$

If we let $p(z) = 1 + \dfrac{zF''_{1,\nu}(z)}{F'_{1,\nu}(z)}$, from (2.4.35) we obtain the

same function (for $\beta = 1$) , i.e.,

$$p(z) + \frac{zp'(z)}{p(z)+\nu} = 1 + \frac{zf''(z)}{f'(z)} \quad.$$

So using similar arguments as used in the proof of Theorem 2.2.1 and Theorem 2.4.1, we obtain the corresponding result for function $f \in K(A,B)$ with respect to the integral transform (2.4.35).

REMARK 2.4.3 : Taking $\beta = 1$, $\nu = 1$ and considering the Libera integral operator $[I_{1,1}(f)](z) = \frac{2}{z} \int_0^z f(t)dt$, from Theorem 2.4.4 and the above remark, we have the following

(i) $I_{1,1}[S^*(A,B)] \subset S^*(\rho_1)$

and

  $I_{1,1}[K(A,B)] \subset K(\rho_1)$

whenever $-1 \leqq B < 0$ and $B < A \leqq -2B$, where $\rho_1 = \rho_1(A,B)$ is given by

(2.4.36) $\rho_1 = \rho(A,B) = 2[F(, \frac{B-A}{B};3 ; \frac{-B}{1-B})]^{-1} - 1,$

(ii) $I_{1,1}[S^*(A,B)] \subset S^*(\rho_2)$

  $I_{1,1}[K(A,B)] \subset K(\rho_2)$

whenever $0 < B < 1$ and $B < A \leqq \min \{2-B, 4B\}$, where $\rho_2 = \rho_2(A,B)$ is given by

  $\rho_2 = 2[F(1, \frac{A-B}{B} ; 3 ; \frac{B}{1+B})]^{-1} - 1$

(iii)   Further for $\beta = 1$, $\nu = 1$ with $A = -B$, $-1 \leqq B < 0$ we obtain

  $I_{1,1}[S^*(-B,B)] \subset S^*(\rho_3)$

and

  $I_{1,1}[K(-B,B)] \subset S^*(\rho_3)$

where ρ_3 obtained from (2.4.36) is given by

$$\rho_3 = \rho_3(-B,B) = 2[F(1,2;3;-B/(1-B)]^{-1} - 1 \cdot$$

(iv) Now taking A $=1-2\alpha$, B $= -1$ with $\alpha \in [\alpha_o,1)$ where

$\alpha_o = \max \{\frac{\beta-\nu-1}{2\beta}, \frac{-\nu}{\beta}\}$ we obtain the main theorem of Mocanu

et al. [85] as a special case of Theorem 2.4.5. Therefore

all the cases considered in [85] follow immediately. So

we omit the details of the particular cases, but it would

be interesting to state the following result for the Libera

integral operator.

If $\alpha \in [-\frac{1}{2},1)$ then for the Libera integral operator

(2.4.1) we have

$$I_{1,1}[S^*(\alpha)] \subset S^*(\rho_4)$$

and

$$I_{1,1}[K(\alpha)] \subset K(\rho_4)$$

where ρ_4 is obtained from (2.4.36) by taking A $= 1-2\alpha$, B $= -1$

and is given by

$$\rho_4 = \rho_4(1-2\alpha,-1) = \begin{cases} \alpha(2\alpha-1)2^{2\alpha-1} - 1, \ \alpha \neq \frac{1}{2}, \ \alpha \neq 0 \\[2mm] \frac{2 \ \ln 2-1}{2 - \ln 2} = 0.6294, \ \alpha = 1/2 \\[2mm] \frac{3-4 \ \ln 2}{4 \ \ln 2-2}, \ \ \alpha = 0. \end{cases}$$

2.6   SOME SUFFICIENT CONDITIONS FOR UNIVALENCE AND STARLIKENESS

Recently, Mocanu [84] showed that for $f \in H$

(2.6.1) $|\frac{f''(z)}{f'(z)}| < \frac{3}{2}$ implies $|\frac{zf'(z)}{f(z)} - 1| < 1$

and in [138] Singh and Singh proved that if for some $\nu \geq 0$,

(2.6.2) $|\frac{zf'(z)}{f(z)} - 1|^{1-\nu} |\frac{zf''(z)}{f'(z)}|^{\nu} < (\frac{3}{2})^{\nu}$

holds then $f \in S^*$. The results (2.6.1) and an improved form
of (2.6.2) can be obtained from the following more general
theorem.

THEOREM 2.6.1 :   If $\delta \geq -1$ and $f \in H$ satisfies the condition

(2.6.3) $|\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - 1|^{1-\nu} |\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} - M|^{\nu} < \beta(M,\delta,\nu), \ z \in U$

for some $\nu \geq 0$ and $M \leq 1$, where $\beta(M,\delta,\nu) \equiv$

$[\frac{(2M(\delta+2)-3)^2 + 8(\delta+1)}{4(\delta+2)^2}]^{\nu/2}$, then

(2.6.4) $|\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - 1| < 1, \ z \in U.$

Proof : Set $p(z) = 2\frac{D^{\delta}f(z)}{D^{\delta+1}f(z)} - 1$, then p is regular in U
with $p(0) = 1$. A simple calculation shows that

$$\left[\frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} - 1\right]^{1-\nu}\left[\frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} - M\right]^{\nu}$$

$$(2.6.5) \quad = \left[\frac{1-p(z)}{1+p(z)}\right]^{1-\nu}\left[\frac{1}{\delta+2}\left\{(\delta+1)\left(\frac{1-p(z)}{1+p(z)}\right) - \frac{zp'(z)}{p(z)+1}\right\} + 1 - M\right]^{\nu}$$

$$\equiv \Psi(p(z), zp'(z)),$$

where

$$\psi(r,s) = \left[\frac{1-r}{1+r}\right]^{1-\nu}\left[\frac{1}{\delta+2}\left\{\frac{\delta+1+(\delta+2)(1-M)-s-r(M(\delta+2)-1)}{1+r}\right\}\right]^{\nu}$$

with $r = p(z)$ and $s = zp'(z)$.

By (2.6.5), we have to prove that

$|\Psi(p(z),zp'(z))| < \beta(M,\delta,\nu)$, $z \in U$ implies that Re $p(z) > 0$ in U which is equivalent to showing (2.6.4). Now for all real r_2, and $s_1 \leq -(1+r_2^2)/2$, we have

$$|\psi(ir_2, s_1)|^2 = \left[\frac{((\delta+1)+(\delta+2)(1-M)-s_1)^2 + r_2^2(M(\delta+2)-1)^2}{1+r_2^2}\right]^{\nu}$$

$$= \frac{1}{(\delta+2)^{2\nu}}\left[\frac{s_1^2 - 2\{(\delta+1)+(\delta+2)(1-M)\}s_1}{1+r_2^2}\right.$$

$$\left. + \frac{4(\delta+1)(\delta+2)(1-M)}{1+r_2^2} + (M(\delta+2)-1)^2\right]^{\nu}$$

$$\geq \frac{1}{(\delta+2)^2}\left[\frac{1}{4} + \{(\delta+1)+(\delta+2)(1-M)\} + (M(\delta+2)-1)^2\right]^{\nu}$$

$$= \left[\frac{(2M(\delta+2)-3)^2 + 8(\delta+1)}{4(\delta+2)^2}\right]^{\nu}$$

$$\equiv \beta(M,\delta,\nu).$$

Taking Ω to be the set $\Omega = \{w \in \mathbb{C} : |w| < \beta(M,\delta,\nu)\}$ we see by Lemma 2.4.1 that Re $p(z) > 0$ in U. Hence the theorem.

REMARK 2.6.1 : Taking $\delta = 0$ and $M = 1$ in Theorem 2.6.1, it follows that for $\nu \geq 0$ and $f \in H$,

$$(2.6.6) \quad \left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\nu} \left|\frac{zf''(z)}{f'(z)}\right|^{\nu} < \left(\frac{3}{2}\right)^{\nu} \text{implies} \left|\frac{zf'(z)}{f(z)} - 1\right| < 1$$

$$z \in U,$$

whereas for $\delta = -1$ and $M = 0$, Theorem 2.6.1 gives

$$(2.6.7) \quad |f'(z)-1|^{1-\nu}\left|1 + \frac{zf''(z)}{f'(z)}\right|^{\nu} < \left(\frac{3}{2}\right)^{\nu} \text{ implies } |f'(z)-1| < 1, z \in U$$

(2.6.6) is an improvement of a result due to Singh and Singh [138, Theorem 3] while (2.6.7) gives a result in [138, Theorem 2]. The case $\nu = 1$ in (2.6.6) reduces to (2.6.1).

ON CERTAIN NEW SUBCLASSES OF UNIVALENT ANALYTIC FUNCTIONS

3.1 INTRODUCTION :

In the previous chapter we studied the properties of the class $T_{\delta,\alpha}(A,B)$ of functions $f \in H$ which satisfy

$$(3.1.1) \quad (1-\alpha) \frac{D^{\delta+1}f(z)}{D^{\delta}f(z)} + \alpha \frac{D^{\delta+2}f(z)}{D^{\delta+1}f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in U$$

where $\alpha \geq 0$, $\delta \geq -1$,

$$(3.1.2) \quad -1 \leq B < 1 \text{ with } B < A, \text{ and}$$

$$(3.1.3) \quad D^{\delta}f(z) = (z/(1-z)^{\delta+1}) * f(z) .$$

In this chapter we first generalize the above subclass of univalent analytic functions in U as follows.

DEFINITION 3.1.1 : Let α be a complex number with Re $\alpha \geq 0$. A function $f \in H$ is said to belong to the class $H_{\delta,\alpha}(h)$ with respect to $g \in H$, $D^{\delta+1}g(z) \neq 0$ in $0 < |z| < 1$, if

$$(3.1.4) \quad (1-\alpha) \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} \prec h(z), \quad (h(0) = 1), z \in U$$

for some $\delta \geq -1$, and h an univalent analytic function in U.

If $h(z) = \frac{1+(1-2\beta)z}{1-z}$ ($\beta < 1$), we denote this class, for convenience, by $H_{\delta,\alpha}^{\beta}$.

Al-Amiri [2] showed that if α is real and positive, $\delta \geq 0$, $f \in H_{\delta,\alpha}^{1/2}$ with respect to g satisfying

(3.1.5) $\quad$ Re $\{\frac{D^{\delta+3}g(z)}{D^{\delta+2}g(z)}\} > \frac{1}{2}$, $z \in U$

then for $\alpha > \beta \geq 0$,

(3.1.6) $\quad$ $H_{\delta,\alpha}^{1/2} \subset H_{\delta,0}^{1/2}$; $H_{\delta,\alpha}^{1/2} \subset H_{\delta,\beta}^{1/2}$.

Very recently, Bulboacă[19] considered the class $H_{n,\alpha}^{\beta}$ for $\delta = n \in N_o$, $\alpha > 0$, $\beta < 1$ with the corresponding g satisfying

(3.1.7) $\quad$ Re $\{\frac{D^{n+3}g(z)}{D^{n+2}g(z)}\} > \beta$, $z \in U$, $\delta = n \in N_o$

and called this class, the class of n-α-close-to-convex functions of order β. Bulboacă[19 , Theorem 1] proved that if $f \in H_{n,\alpha}^{\beta}$ ($\alpha \geq 0$, $\frac{1}{2} \leq \beta < 1$ and $n \in N_o$) with respect to g satisfying (3.1.7) then $f \in H_{n,0}^{\beta^*}$ with respect to g satisfying Re $\{\frac{D^{n+2}g(z)}{D^{n+1}g(z)}\} > \beta^*$ ($z \in U$), where $\beta^* = [F(1,2(n+3)(1-\beta);n+3;\frac{1}{2})]^{-1}$, F being defined by (2.2.2). It may be noted that for $\beta = \frac{1}{2}$ we obtain $\beta^* = \frac{1}{2}$ and so the above result reduces to (3.1.6).

Further it can be seen that for $g(z) = z$ and $\delta = -1$, ($\alpha \geq 0$)(3.1.6) reduces to

Re $\{(1-\alpha) \frac{f(z)}{z} + \alpha f'(z)\} > \frac{1}{2}$ implies Re $\{\frac{f(z)}{z}\} > \frac{1}{2}$, $z \in U$,

a result analogous to a result of Chichra [24], who showed
that for $f \in H$, Re $\{(1-\alpha) \frac{f(z)}{z} + \alpha f'(z)\} > 0$ implies

Re $\{\frac{f(z)}{z}\} > 0$, $z \in U$. It may be noted further that it is
not possible to obtain sharp result from Al-Amiri and
Bulboca's results.

In this chapter we first give, in Section 3.2,
containment relations for the classes $H_{\delta,\alpha}(h)$ and $H_{\delta,\alpha}^{\beta}$ which
generalize and improve many of the earlier known results
obtained by Al-Amiri, Bulboca and Chichra. These relations
also give rise to a number of new results as particular cases.
In Section 3.3, a study of Bernardi transform of functions
in $H_{\delta,0}(h)$ leads to improvement and generalizations of a
number of known results including those of Libera [61],
Singh and Singh [137], Bernardi [8], Pascu [94], Al-Amiri
[1,2], Bulboacă [19] and others. Some applications to
hypergeometric functions find a place in Section 3.4. The
results in Section 3.4 improve and generalize the earlier
results of Owa and Srivastava [90].

3.2 CONTAINMENT RELATIONS :

For the proof of our results we need the following
lemma.

LEMMA 3.2.1 : Let h be a convex univalent function in U,
with h(0) = p(0) = c and let $\lambda(z)$ be an analytic function

<u>in</u> U <u>with</u> Re $\{\lambda(z)\} > 0$. <u>If</u> $p(z) = c + p_1 z + \ldots$ <u>is</u> <u>analytic</u>
<u>in</u> U, <u>and</u> <u>satisfies</u> <u>the</u> <u>differential</u> <u>subordination</u>

(3.2.1) $p(z) + zp'(z) (\lambda(z)) \prec h(z)$, $z \in U$

<u>then</u> $p(z) \prec h(z)$, $z \in U$.

<u>Proof</u> : Let us first suppose that all the functions under
consideration are analytic in the closed disc $\bar{U}$. For that
we shall first show that if $p(z)$ is not subordinate to $h(z)$,
then there is a z_0, $z_0 \in U$, such that

(3.2.2) $p(z_0) + z_0 p'(z_0) (\lambda(z_0)) \notin h(U)$

which would contradict the hypothesis.

        If $p(z)$ is not subordinate to $h(z)$, then, by Lemma
1.5.2, we conclude that there are $z_0 \in U$, $\zeta_0 \in \partial U$, and m,
$m \geq 1$, such that

(3.2.3) $p(z_0) + z_0 p'(z_0)\lambda(z_0) = h(\zeta_0) + m\zeta_0 h'(\zeta_0) \lambda(z_0)$.

Now   Re $\{\lambda(z)\} > 0$ in U implies $|\arg(\lambda(z))| < \pi/2$,
and $\zeta_0 h'(\zeta_0)$ is in the direction of the outer normal to
the convex domain $h(U)$, so that the right-hand member of
(3.2.3) is a complex number outside $h(U)$; that is, (3.2.2)
holds. Because this contradicts the hypothesis namely (3.2.1),
we conclude that $p(z) \prec h(z)$, provided all functions under
consideration are analytic in $\bar{U}$.

        To remove this restriction, we need but replace $p(z)$
by $p_\rho(z) = p(\rho z)$ and $h(z)$ by $h_\rho(z) = h(\rho z)$, $0 < \rho < 1$. All

the hypotheses of the theorem are satisfied, and we conclude that $p_\rho(z) \prec h_\rho(z)$ for each ρ, $0 < \rho < 1$. By letting $\rho \to 1$, we obtain $p(z) \prec h(z)$ in U.

REMARK 3.2.1 : The lemma above proves the existence of a dominant for the differential subordination (3.2.1). It seems plausible that each $p(z)$ satisfying the differential subordination (3.2.1) in Lemma 3.2.1, will have best dominant, of course with an additional condition on $\lambda(z)$. However, we have not been able to prove it at present. It may be noted that for the case $\lambda(z) = \frac{1}{\nu}$ where ν is a complex number with $\nu \neq 0$, Re $\nu \geq 0$ in (3.2.1), the best dominant of (3.2.1) is known [44, 75] (see Lemma 2.2.4). If we take $p(z) = 1+z$, $\lambda(z) = \frac{1+z}{1-z}$ in the above lemma then it can be seen that

$$p(z) + zp'(z)\,(\lambda(z)) = \frac{1+z}{1-z}$$

and so

Re $\{p(z) + zp'(z)\,(\lambda(z))\} > 0$ implies Re $p(z) > 0$ in U

follows from Lemma 3.2.1. This shows that the Lemma 3.2.1 may be improved with an additional condition on $\lambda(z)$.
We now proceed to prove

THEOREM 3.2.1 : Let $\delta \geq -1$, α be the complex number with Re $\alpha > \eta \geq 0$, and $g \in H$ satisfies

$$(3.2.4) \qquad \text{Re}\,\{\alpha\,\frac{D^{\delta+1}g(z)}{D^{\delta+2}g(z)}\} > \eta, \quad z \in U.$$

(a) Then with respect to this g and h, a convex univalent function with $h(0) = 1$, we have

(3.2.5) $H_{\delta,\alpha}(h) \subset H_{\delta,0}(h)$ whenever $\eta = 0$

and

(3.2.6) $H_{\delta,\alpha}^{\beta} \subset H_{\delta,0}^{\beta_1}$, whenever $\eta > 0$

where

(3.2.7) $\beta_1 \equiv \dfrac{2\beta(\delta+2) + \eta}{2(\delta+2) + \eta} \equiv \beta + \dfrac{\eta(1-\beta)}{2(\delta+2)+\eta}$.

(b) If α be a real number with $\alpha \geq 1$ and $g \in H$ satisfies (3.2.4), then

(3.2.8) $H_{\delta,\alpha}^{\beta} \subset H_{\delta,1}^{\beta_2}$

where

(3.2.9) $\beta_2 \equiv \beta + \dfrac{\eta(\alpha-1)(1-\beta)}{\alpha(2(\delta+2)+\eta)}$.

Proof : Let $f \in H_{\delta,\alpha}(h)$, where $\delta \geq -1$, α is a complex number satisfying $\text{Re } \alpha > 0$, h be a convex univalent function with $h(0) = 1$ and $g \in H$ satisfies (3.2.4) with $\eta = 0$. Set $\lambda(z) = \alpha \dfrac{D^{\delta+1}g(z)}{D^{\delta+2}g(z)}$. By (3.2.4) we observe that $\text{Re }\{\lambda(z)\} > 0$, $\text{Re } \alpha > 0$ and $D^{\delta+1}g(z) \neq 0$, in $0 < |z| < 1$. Define

(3.2.10) $p(z) = \dfrac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}$.

Clearly $p(z)$ is analytic in U, $p(0) = 1$. Differentiating (3.2.10) and using the identity (2.2.14), we get

$$(3.2.11) \quad \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} = p(z) + \frac{1}{\delta+2} \frac{D^{\delta+1}g(z)}{D^{\delta+2}g(z)} zp'(z).$$

Therefore, from (3.2.10) and (3.2.11), we obtain

$$(1-\alpha) \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} = p(z) + (\frac{\alpha}{\delta+2} \frac{D^{\delta+1}g(z)}{D^{\delta+2}g(z)}) zp'(z)$$

$$(3.2.12) \qquad\qquad = p(z) + \lambda_1(z) zp'(z)$$

where $\lambda_1(z) = \frac{1}{\delta+2} (\lambda(z))$ satisfies Re $\{\lambda_1(z)\} > 0$.

Since $f \in H_{\delta,\alpha}(h)$, we obtain from (3.2.12) that

$$(3.2.13) \qquad p(z) + \lambda_1(z) zp'(z) \prec h(z).$$

Now using Lemma 3.2.1, we obtain from (3.2.13), that

$$(3.2.14) \qquad\qquad p(z) \prec h(z).$$

(3.2.14), because of (3.2.10), gives that $f \in H_{\delta,0}(h)$. This proves (3.2.5).

To prove (3.2.6), let $f \in H_{\delta,\alpha}^{\beta}$ where $\delta \geq -1$, $\beta < 1$, α is a complex number satisfying Re $\alpha > \eta > 0$ and $g \in H$ satisfying (3.2.4) with $\eta > 0$. By (3.2.4) we again observe that Re $\{\lambda(z)\} > \eta, 0 < \eta <$ Re α and $D^{\delta+1}g(z) \neq 0$ in $0 < |z| < 1$. Define

$$(3.2.15) \qquad p(z) = (1-\beta_1)^{-1} (\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} - \beta_1)$$

where β_1 is given by (3.2.7). Clearly p(z) is analytic in U, p(O) = 1. To prove (3.2.6), it is sufficient to show that $f \in H^{\beta}_{\delta,\alpha}$ implies Re p(z) > O in U. Differentiating (3.2.15) and using (2.2.14), we get

$$(1-\alpha) \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} = \beta_1+(1-\beta_1)p(z)+\frac{(1-\beta)}{\delta+2} (\lambda(z)).zp'(z)$$

(3.2.16) $\equiv \Psi(p(z),zp'(z);z)$, say

where

(3.2.17) $\Psi(r,s;z) = \beta_1 + (1-\beta_1)r + \dfrac{(1-\beta_1)}{\delta+2} (\lambda(z)) s.$

Since $f \in H^{\beta}_{\delta,\alpha}$ we obtain

(3.2.18) $\{\Psi(p(z),zp'(z);z):z \in U\} \subset \Omega \equiv \{w \in \mathbb{C}:\text{Re } w > \beta\}$.

Now we show that for each $z \in U$, Ψ satisfies the condition (2.4.28) of Lemma 2.4.1 with k = 1 and a = 1. For all real r_2, and $s_1 \leqq \dfrac{-(1+r_2^2)}{2}$, we have from (3.2.17) that

$$\text{Re } \left\{ \Psi(ir_2,s_1;z)\right\} = \beta_1 + \frac{(1-\beta_1)}{\delta+2} s_1 \text{ Re}(\lambda(z))$$

$$\leqq \beta_1 - \frac{(1-\beta_1)}{2(\delta+2)} \eta \equiv \beta.$$

This shows that for each $z \in U$, $\Psi(ir_2,s_1;z) \notin \Omega$.
Hence by (3.2.18) and Lemma 2.4.1, we obtain that Re p(z) > O in U. This by (3.2.15) shows that (3.2.6) holds true.

Case (b) : If $f \in H_{\delta,\alpha}^{\beta}$ with $\beta < 1$, $\delta \geq -1$ and α be a real number satisfying $\alpha \geq 1$, we have from (3.2.6) that

$$(1-\alpha) \text{ Re } \{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\} + \alpha \text{ Re } \{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\} > \beta, \quad z \in U$$

implies Re $\{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\} > \beta_1$, $z \in U$.

Since $\alpha \geq 1$, it follows that

$$(3.2.19) \qquad (1-\alpha) \text{ Re } \{\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}\} \leq \beta_1(1-\alpha).$$

Since $f \in H_{\delta,\alpha}^{\beta}$, (3.2.19) leads to

$$(3.2.20) \quad \text{Re } \{\frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\} > \frac{\beta-\beta_1(1-\alpha)}{\alpha}, \quad z \in U.$$

Now (3.2.9) follows upon substituting (3.2.7) in (3.2.20). Hence the theorem.

REMARK 3.2.2 : Though we are not able to obtain the sharpness of the above theorem, but for the special case namely $g(z) = z$, the result has been shown to be sharp in Theorem 2.2.2. Considering α real with $\alpha \geq 0$ and $\eta = 0$ in the above theorem we obtain

COROLLARY 3.2.1 : Let h be a convex univalent function in U, with $h(0) = 1$. Then for α real and $\alpha \geq 0$,

$$H_{\delta,\alpha}(h) \subset H_{\delta,0}(h)$$

<u>whenever</u> $\text{Re} \{ \dfrac{D^{\delta+2}g(z)}{D^{\delta+1}g(z)} \} > 0.$

It may be noted that the above corollary not only generalizes the result of Al-Amiri [12] and Bulboacă[19] but also shows that for $h(z) = \dfrac{1}{1-z}$ the containment relation (3.1.6) of Al-Amiri [2] holds under much weaker hypothesis on g and similarly for $h(z) = \dfrac{1+(1-2\beta)z}{1-z}$ $(\tfrac{1}{2} \le \beta < 1)$ the containment relation of Bulboacă[19] holds under weaker condition on g.

<u>COROLLARY 3.2.2</u> : <u>Let</u> $\delta \ge -1$, $\alpha > \alpha' \ge 0$ <u>and</u> h <u>a</u> <u>convex</u> <u>univalent</u> <u>function</u> <u>in</u> U. <u>If</u> $f \in H_{\delta,\alpha}(h)$ <u>with</u> <u>respect</u> <u>to</u> $g \in H$ <u>satisfying</u> (3.2.4) <u>with</u> $\eta = 0$ <u>then</u> $f \in H_{\delta,\alpha'}(h)$ <u>with</u> <u>respect</u> <u>to</u> <u>the</u> <u>same</u> g.

<u>Proof</u> : If $\alpha' = 0$, then there is nothing to prove. If $\alpha' > 0$ and $f \in H_{\delta,\alpha}(h)$ with respect to $g \in H$ satisfying (3.2.4) for α real with $\alpha > \alpha' > 0$, then from the identity

$$(3.2.21) \quad (1-\alpha') \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha' \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}$$

$$= \frac{\alpha'}{\alpha}[\ (1-\alpha)\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha \frac{D^{\delta+2}f(2)}{D^{\delta+2}g(z)}\ ] + (1-\frac{\alpha'}{\alpha})\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)}]$$

and the fact that $f \in H_{\delta,\alpha}(h)$ implies $f \in H_{\delta,0}(h)$,(cf.(3.2.5)), we see that

$$(1-\alpha')\frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha' \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)} \prec \frac{\alpha'}{\alpha}h(z) + (1-\frac{\alpha'}{\alpha})h(z) \equiv h(z), z \in U.$$

This shows that $f \in H_{\delta, \alpha'}(h)$.

COROLLARY 3.2.3 : Let $\delta \geq -1$ and $\alpha > \max \{\alpha', \eta\} \geq 0$. If $f \in H_{\delta, \alpha}^{\beta}$ with respect to $g \in H$ satisfying (3.2.4) with $\eta \geq 0$ then $f \in H_{\delta, \alpha}^{\beta_2}$ with respect to the same g, where

$$\beta_2 \equiv \beta + (1 - \frac{\alpha'}{\alpha}) \frac{\eta(1-\beta)}{2(\delta+2)+\eta} .$$

Proof : The cases $\beta = 0$ and $\eta = 0$ are obvious, hence assume $\beta > 0$, and $\eta > 0$.

Since β_1 defined by (3.2.7) is greater than or equal to β, from the identity (3.2.21) and (3.2.6), we obtain

$$\text{Re } \{(1-\alpha') \frac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} + \alpha' \frac{D^{\delta+2}f(z)}{D^{\delta+2}g(z)}\} > \frac{\alpha'}{\alpha}\beta + (1-\frac{\alpha'}{\alpha})\beta_1 = \beta + (1-\frac{\alpha'}{\alpha})\frac{\eta(1-\beta)}{2(\delta+2)+\eta}$$

$$\equiv \beta_2.$$

Hence the corollary.

The case $\delta = -1$ of Theorem 3.2.1 gives the following

COROLLARY 3.2.4 : If $g \in H$ satisfies

$$\text{Re } (\alpha \frac{g(z)}{zg'(z)}) > \eta ,$$

α be a complex number with Re $\alpha > \eta \geq 0$ and $f \in H$. Then

(a) for $\eta \geq 0$ and h a convex univalent function in U,

(3.2.22) $(1-\alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} \prec h(z), z \in U$

implies $\frac{f(z)}{g(z)} \prec h(z), \quad z \in U.$

(b) <u>For</u>, $\beta < 1$, $\eta > 0$ <u>and</u> α <u>real</u>

(3.2.23) Re $\{(1-\alpha) \dfrac{f(z)}{g(z)} + \alpha \dfrac{f'(z)}{g'(z)}\} > \beta$, $z \in U$

<u>implies</u> Re $\{\dfrac{f(z)}{g(z)}\} > \dfrac{2\beta+\eta}{2+\eta}$, $z \in U$.

(c) <u>for</u> α <u>a real number satisfying</u> $\alpha \geq 1$, <u>and</u> $\eta > 0$ <u>we have</u> (3.2.23) <u>implies</u>

(3.2.24)   Re $\{\dfrac{f'(z)}{g'(z)}\} > \beta + \dfrac{\eta(\alpha-1)(1-\beta)}{\alpha(2+\eta)}$, $z \in U$.

For $\alpha = 1$, the above corollary shows that if $g \in H$ satisfies Re $\{\dfrac{g(z)}{zg'(z)}\} > \eta$, $0 \leq \eta < 1$, then

(3.2.25) Re $\{\dfrac{f'(z)}{g'(z)}\} > \beta$ implies Re $\{\dfrac{f(z)}{g(z)}\} > \dfrac{2\beta+\eta}{2+\eta} \geq \beta$, $z \in U$.

The relation (3.2.25) yields the results of MacGregor [67] and Libera [61] for $\eta = 0$. However if $\eta > 0$, (3.2.5) gives an improvement over their results.

REMARK 3.2.3 : It is interesting to observe that if $g(z) = z/(1+z)^2$ (and hence g satisfies Re$(zg'(z)/g(z)) > 0$ in U) and f be determined by $f'(z)/g'(z) = (1+(1-2\beta)z)/(1-z)$ $(\beta < 1)$ then $f(z)/g(z) = (1-\beta)(1+z)+\beta$. This shows that the bound in the relation (3.2.25) cannot be improved for the case $\eta = 0$, thereby establishing that the results of MacGregor [67] and Libera [61] are the best possible ones.

3.3   INTEGRAL TRANSFORMS :

For functions f ∈ H and g ∈ H, Libera [61] showed that

(3.3.1)   Re $\{\frac{f'(z)}{g'(z)}\}$ > O implies Re $\{\frac{[I_{1,1}(f)]'(z)}{[I_{1,1}(g)]'(z)}\}$ > O when g ∈ K

and in [8], Bernardi showed that

(3.3.2)   Re $\{\frac{f'(z)}{g'(z)}\}$ > O implies Re $\{\frac{[I_{1,c}(f)]'(z)}{[I_{1,c}(g)]'(z)}\}$ > O when g ∈ K

and c ∈ N, where $I_{1,c}(f)$ is defined by

(3.3.3) $[I_{1,c}(f)](z) = \frac{1+c}{z^c} \int_O^z t^{c-1} f(t)dt.$

Pascu [94] showed that (3.3.2) continues to hold for any real c > -1. On the other hand, Singh and Singh [137] showed that (3.3.1) is valid even if g ∈ K(-1/2).

Mocanu [84 , Theorem 2] proved the following lemma.

LEMMA 3.3.1 :   Let $Q_{\mu+c}(z)$ be the unique function that maps U onto the complex plane slit along the half lines Re w = O,

$|$ Im w $| \geqq \frac{1}{Re(\mu+c)}$ [$|\mu+c|\sqrt{1+2\ Re(\mu+c)}-Im(\mu+c)$], $(Q_{\mu+c}(O)=\mu+c$ with

Re (μ+c) > O).

If g ∈ H satisfies the subordination relation

$$\mu\,\frac{zg'(z)}{g(z)} + c \prec Q_{\mu+c}(z)$$

then the function $G = I_{\mu,c}(g)$ defined by

$$[I_{\mu,c}(g)](z) = G(z) = [\frac{\mu+c}{z^c} \int_0^z t^{c-1} g^{\mu}(t)dt]^{1/\mu} \text{ is analytic}$$

in U, $G(z)/z \neq 0$ and Re $(\mu z \frac{G'(z)}{G(z)} + c) > 0$ in U.

It can be easily seen that

$$(3.3.4) \quad D^{\delta+1}([I_{1,c}(g)](z)) = \frac{1+c}{z^c} \int_0^z t^{c-1} D^{\delta+1} g(t) \, dt$$

and so the above lemma continues to hold for $\mu = 1$ and $g(z)$ and $[I_{1,c}(g)](z)$ replaced by $D^{\delta+1}g(z)$ and $D^{\delta+1}([I_{1,c}(g)](z))$ respectively. We now state and prove the following theorem.

THEOREM 3.3.1 : Let c be a complex number with Re$(1+c) > 0$, $\delta \geq -1$ and h be a convex univalent function in U with $h(0) = 1$. If $g \in H$ and

$$(3.3.5) \qquad z \frac{(D^{\delta+1} g(z))'}{D^{\delta+1} g(z)} + c \prec Q_{1+c}(z)$$

then

$$(3.3.6) \qquad f \in H_{\delta,0}(h) \text{ with respect to } g$$

implies

$$(3.3.7) \qquad F \in H_{\delta,0}(h) \text{ with respect to } G$$

where $F(z) = [I_{1,c}(f)](z)$ and $G(z) = [I_{1,c}(g)](z)$ are defined by (3.3.3) and $Q_{1+c}(z)$ is as defined in Lemma 3.3.1.

Proof : Since $[I_{1,c}(f)](z) \equiv (\sum_{j=1}^{\infty} \frac{c+1}{c+j} z^j) * f(z) = F(z)$, (say) and

$$[I_{1,c}(g)](z) = (\sum_{j=1}^{\infty} \frac{c+1}{c+j} z^j) * g(z) \equiv G(z), \text{ say,}$$

it can be easily seen that

(3.3.8) $\qquad z(D^{\delta+1}F(z))' = (1+c)D^{\delta+1}f(z) - c\ D^{\delta+1}F(z),$

and

(3.3.9) $\qquad z(D^{\delta+1}G(z))' = (1+c)\ D^{\delta+1}g(z) - c\ D^{\delta+1}G(z).$

Set

(3.3.10) $\qquad \lambda(z) = [z\ \dfrac{(D^{\delta+1}G(z))'}{D^{\delta+1}G(z)} + c]^{-1}$.

Since $g \in H$ satisfies (3.3.5), it follows from Lemma 3.3.1 that $D^{\delta+1}G(z)$ defined by (3.3.4) is analytic in U, $D^{\delta+1}G(z)/z \neq 0$ and $\operatorname{Re} \{z\dfrac{(D^{\delta+1}G(z))'}{D^{\delta+1}G(z)} + c\} > 0$ in U and this in turn implies that $\operatorname{Re} \lambda(z) > 0$ in U.

Now consider

(3.3.11) $\qquad p(z) = \dfrac{D^{\delta+1}F(z)}{D^{\delta+1}G(z)}$.

Then p(z) is analytic in U and p(0) = 1. By (3.3.8) and (3.3.9), a simple manipulation shows that

(3.3.12) $\qquad \dfrac{D^{\delta+1}f(z)}{D^{\delta+1}g(z)} = p(z) + (\lambda(z))\ zp'(z)$

where $\lambda(z)$ in (3.3.12) is given by (3.3.10) and so $\operatorname{Re} \lambda(z) > 0$ in U.

Since $f \in H_{\delta,0}(h)$ with respect to g and h be convex univalent function in U with h(0) = 1, it follows from (3.3.12) and Lemma 3.2.1 that $p(z) \prec h(z)$ in U and hence

by (3.3.11), we obtain the required conclusion. Hence the theorem.

COROLLARY 3.3.1 : Taking $\delta = 0$ in Theorem 3.3.2, it follows that if $f \in H$, h convex univalent function in U with $h(0) = 1$ and $g \in H$ satisfies

$$1+c + \frac{zg''(z)}{g'(z)} \prec Q_{1+c}(z), \quad z \in U \quad (Re \ (1+c) > 0),$$

then

$$\frac{f'(z)}{g'(z)} \prec h(z) \quad \underline{implies} \quad \frac{[I_{1,c}(f)]'(z)}{[I_{1,c}(g)]'(z)} \prec h(z) \quad z \in U$$

where $[I_{1,c}(f)](z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)dt$ and $Q_{1+c}(z)$ is the unique function that maps U onto the complex plane slit along the half lines

$$Re \ w = 0, \quad |Im(w)| \geq \frac{1}{Re(1+c)} \ [|1+c| \ \sqrt{1+2 \ Re(1+c)} - Im(1+c)]$$

$(Q_{1+c}(0) = 1+c)$.

The above corollary for $h(z) = \frac{1+(1-2\beta) \ z}{1-z}$ shows that for $f, g \in H$ and $\beta < 1$

$$(3.3.13) \quad Re \ \{\frac{f'(z)}{g'(z)}\} > \beta \quad implies \quad Re \ \{\frac{[I_{1,c}(f)]'(z)}{[I_{1,c}(g)]'(z)}\} > \beta, z \in U.$$

whenever $g \in H$ satisfies $1+c + \frac{zg''(z)}{g'(z)} \prec Q_{1+c}(z)$

(3.3.13) shows that result (3.3.1) of Libera holds under much weaker hypotheses than the one considered by Singh and Singh [137] for the case $c = 1$, namely $g \in K(-\frac{1}{2})$.

In particular from (3.3.13) we also get

$$\text{Re } \{\frac{f'(z)}{g'(z)}\} > \beta \text{ implies Re } \{\frac{[I_{1,c}(f)]'(z)}{[I_{1,c}(g)]'(z)}\} > \beta, \; z \in U$$

whenever $g \in K(-\text{Re}(c))$.

This is also an improved and generalized form of the results obtained by Bernardi [8], Pascu [94] and others.

3.4 SOME APPLICATIONS TO GENERALIZED HYPERGEOMETRIC FUNCTIONS:

Let a_i (i = 1,2,...,p) and b_i (i = 1,2,...,q) be complex numbers with $b_i \neq 0, -1, -2, ...$ (i = 1,2,...,q). Then the generalized hypergeometric function $_pF_q(z)$ is defined by

$$_pF_q(z) \equiv \; _pF_q (a_,,...,a_p; \; b_1,...,b_q;z)$$

$$(3.4.1) \qquad = \sum_{n=0}^{\infty} \frac{(a_1)_n \; ... \; (a_p)_n}{(b_1)_n \; ... \; (b_q)_n} \; \frac{z^n}{n!} \quad (p \leq q+1)$$

where the notation $(a)_n$ means the Poch hammer symbol defined by

$$(a)_n = \begin{cases} 1 & , \text{ if } n = 0 \\ \\ a(a+1) \; ... \; (a+n-1), & \text{ if } n \in N = \{1,2,...\} \end{cases}.$$

We note that $_pF_q$ series in (3.4.1) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in U$, if $p = q+1$.

The function $_2F_1(a_1,a_2;b_1;z)$ is the hypergeometric function and $z_2F_1(1,a_2;b_1;z)$ is the incomplete beta function

given conveniently by

$$\varphi(a,b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} z^{n+1}, \quad |z| < 1, \quad b \neq 0, -1, -2, \ldots,$$

and having an analytic continuation to the z-plane cut along the positive real axis. In [90], Owa and Srivastava [Theorem 4 and Corollary 3] proved that

$$(3.4.2) \quad z_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) * (z/(1-z)^2) \in H^\beta_{-1,0}$$

with respect to $g \in S^*$

implies

$$(3.4.3) \quad z_{p+k}F_{q+k}(a_1,\ldots,a_p,c_1+1,\ldots,c_k+1;b_1,\ldots,b_q,$$

$$c_1+2,\ldots,c_k+2;z) * (z/(1-z)^2) \in H^\beta_{-1,0}$$

with respect to $z \; {}_{p+k}F_{q+k}(a_1,\ldots,a_p,c_1+1,\ldots,c_k+1;$

$$b_1,\ldots,b_q,c_1+2,\ldots,c_k+2;z) * g(z)$$

where $c_j \geqq 0$ $(j = 1,2,\ldots,k)$.

In the following theorem we improve and generalize the above result as follows:

THEOREM 3.4.1 : Let δ be a real number satisfying $\delta \geq -1$, c_j $(j = 1,2,\ldots,k)$ be complex numbers such that Re $(1+c_j) > 0$ for $j = 1,2,\ldots,k$ and h be a convex univalent function in U with $h(0) = 1$. Further for $f,g \in H$, let φ and Ψ be defined by

(3.4.4) $\qquad \varphi(z) = z \; {}_pF_q(z) * f(z)$

(3.4.5) $\qquad \Psi(z) = z \; {}_pF_q(z) * g(z)$

and Q_{1+c_1} be the unique function that maps U $(p \leq q+1)$ onto the complex plane slit along the half lines

$\text{Re } w = 0, \; |\text{Im } w| \geqq \dfrac{1}{\text{Re}(1+c_1)} \left[|1+c_1| \; \sqrt{1+2 \; \text{Re}(1+c_1)} - \text{Im}(c_1) \right].$

if Ψ satisfies

(3.4.6) $\qquad c_1 + \dfrac{z(D^{\delta+1} \; \Psi(z))'}{D^{\delta+1} \; \Psi(z)} \prec Q_{1+c_1}(z), \; z \in U$

then

$$\varphi \in H_{\delta,0}(h) \text{ with respect to } \Psi$$

implies

$$z \; {}_{p+k}F_{q+k}(z) * f(z) \in H_{\delta,0}(h) \text{ with respect to}$$

$$z \; {}_{p+k}F_{q+k}(z) * g(z),$$

where

(3.4.7) $\quad z \; {}_{p+k}F_{q+k}(z) = z \; {}_{p+k}F_{q+k}(a_1,\ldots,a_p,c_1+1,\ldots,c_k+1;$

$$b_1,\ldots,b_q,c_1+2,\ldots,c_k+2;z).$$

Proof : Consider the integral operator

$$[I_{1,c_1}(\varphi)](z) = \dfrac{1+c_1}{z^{c_1}} \int_0^z t^{c_1-1} \; \varphi(t) \; dt$$

$$= \dfrac{1+c_1}{z^{c_1}} \int_0^z t^{c_1-1}[t \; {}_pF_q(t) * f(t)] \; dt$$

$$= \frac{1+c_1}{z^{c_1}} \int_0^z t^{c_1-1} [\{t_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\}*(t+\sum_{n=2}^{\infty} a_n t^n)]dt$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{c_1+1}{n+c_1+1} a_{n+1} \frac{z^{n+1}}{n!} \quad (a_1 = 1)$$

$$= [\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n (c_1+1)_n}{(b_1)_n \cdots (b_q)_n (c_1+2)_n} \frac{z^{n+1}}{n!}] * f(z)$$

$$= [z_{p+1}F_{q+1}(a_1,\ldots,a_p,c_1+1;b_1,\ldots,b_q,c_1+2;z)] * f(z).$$

Similarly we get

$$[I_{1,c_1}(\Psi)](z) = [z \, _{p+1}F_{q+1}(a_1,\ldots,a_p,c_1+1;b_1,\ldots,b_q,c_1+2;z)]*g(z)$$

Since Ψ satisfies (3.4.6), we obtain from Theorem 3.3.1, that

$$\varphi \in H_{\delta,0}(h) \text{ with respect to } \Psi$$

implies $z \, _{p+1}F_{q+1}(z) * f(z)$ is also in $H_{\delta,0}(h)$ with respect to $z \, _{p+1}F_{q+1}(z) * g(z)$.

Again (3.4.6) and Theorem 3.3.1 implies that

$$(3.4.8) \quad \text{Re}[c_1 + \frac{z(D^{\delta+1}([I_{1,c_1}(\Psi)]'(z))}{D^{\delta+1}([I_{1,c_1}(\Psi)](z))}] > 0.$$

Now considering

$$[I_{1,c_2}((z \, _{p+1}F_{q+1}) * f)](z) = \frac{1+c_2}{z^{c_2}} \int_0^z t^{c_2-1}[t_{p+1}F_{q+1}(t)*f(t)]dt$$

where $z \,_{p+1}F_{q+1}(z)$ is defined by (3.4.7). Now using the above arguments we obtain,

$$[I_{1,c_2}(z \,_{p+1}F_{q+1}) * f](z) = [z \,_{p+2}F_{q+2}(z)] * f(z)$$

and

$$[I_{1,c_2}(z \,_{p+1}F_{q+1}) * g](z) = [z \,_{p+2}F_{q+2}(z)] * g(z).$$

This shows the $z \,_{p+2}F_{q+2}(z) * f(z)$ is also in $H_{\delta,0}(h)$ with respect to $z \,_{p+2}F_{q+2}(z) * g(z)$ follows from Theorem 3.3.1 and the fact that $\varphi \in H_{\delta,0}(h)$ with respect to Ψ implies $[z \,_{p+1}F_{q+1}(z)] * f(z)$ is also in $H_{\delta,0}(h)$ with respect to $[z \,_{p+1}F_{q+1}(z)] * g(z)$.

Now repeating the arguments in this fashion, by using Theorem 3.3.1 and considering the operator

$$I_{1,c_i}((z \,_{p+i}F_{q+i}) * f)(z) \qquad (i = 3,4,\ldots,k),$$ yield the required conclusion immediately.

Substituting $\delta = 0$ and taking $f(z) = z/(1-z)$ in the above theorem we obtain

<u>COROLLARY 3.4.1</u> : <u>Let</u> h <u>be convex univalent function with</u> $h(0) = 1$ <u>and</u> Ψ <u>defined by</u> (3.4.5) <u>satisfy</u>

$$c_1 + \frac{z(z \,\Psi'(z))'}{\Psi'(z)} \prec Q_{1+c_1}(z)$$

<u>where</u> Q_{1+c_1} <u>is as defined in Theorem</u> 3.4.1.
<u>Then for</u> $\mathrm{Re}\,(1+c_j) > 0$, $(j = 1,2,\ldots,k)$,

$$\frac{(z \,_pF_q)'(z)}{\Psi'(z)} \prec h(z) \quad \underline{implies} \quad \frac{(z_{p+k}F_{q+k})'(z)}{(z \,_{p+k}F_{q+k} * g)'(z)} \prec h(z), \ z \in U.$$

Corollary 3.4.1 improves and generalize the results of Owa and Srivastava [90],namely, (3.4.2) and (3.4.3).

Taking $g(z) = z$, i.e. $\Psi(z) \equiv z$, we obtain

COROLLARY 3.4.2 : If the function $z \, {}_pF_q$ $(p \leq q+1)$ defined by (3.4.1) satisfies

$$(z \, {}_pF_q)'(z) \prec h(z) \text{ implies } (z \, {}_{p+k}F_{q+k})'(z) \prec h(z), \ z \in U$$

where h is convex univalent function with $h(0) = 1$ and $z \, {}_{p+k}F_{q+k}$ is as defined by (3.4.7) with Re $(1+c_j) > 0$ $(j = 1,2,\ldots,k)$.

It may be noted from the above corollary that for $h(z) = (1+z)/(1-z)$, $z \, {}_pF_q$ close-to-convex (univalent) implies that $z \, {}_{p+k}F_{q+k}$ is also close-to-convex (univalent) in U.

CHAPTER - IV

DIFFERENTIAL SUBORDINATION AND CONFORMAL MAPPINGS

4.1 INTRODUCTION : For $|\lambda| < \pi/2$, let $S^\lambda(\beta)$ denote the class of functions $f \in H$ satisfying

$$(4.1.1) \qquad \text{Re } \{e^{i\lambda} \frac{zf'(z)}{f(z)}\} > \beta \cos \lambda, \quad z \in U,$$

where $\beta < 1$. Then f is said to be λ-spiral-like of order β. As shown by Špaček [141] λ-spiral-like functions are univalent in U. The concept of order for λ-spiral-like functions was introduced by Libera [62], who also made a detailed study of $S^\lambda(\beta)$. The class of 0-spiral-like functions of order β, is the well-known class of starlike functions of order β, denoted by $S^*(\beta)$.

Let g be a normalized starlike function in U, h be analytic function in U such that $h(0) = 1$ and Re $\{e^{i\nu} h(z)\} > 0$ for some real ν and all $z \in U$. Then if $\alpha > 0$ and β real, the function f defined by

$$(4.1.2) \qquad f(z) = [\int_0^z g^\alpha(t)h(t)t^{i\beta-1} dt]^{1/(\alpha+i\beta)}$$

is single-valued, analytic and univalent in U [7,100,125] and is called Bazilevič function of type (α,β). (Powers in (4.1.2) are taken to be principal values). We shall denote by $M(\alpha+i\beta)$, the class of Bazilevič of type (α,β). Then f

is in $M(\alpha+i\beta)$ if and only if there exists a function $g \in S^*$ and a real ν such that

$$(4.1.3) \qquad \text{Re } \{\frac{e^{i\nu} f(z)^{\alpha+i\beta-1} f'(z)}{g^{\alpha}(z) z^{i\beta-1}}\} > 0, \; z \in U.$$

In [111], St. Ruscheweyh proved that if f be Bazilevič of type (α,β) in U and c be a complex number such that Re $c \geq 0$, then the function F defined by

$$(4.1.4) \qquad F(z) = [\frac{c+\alpha+i\beta}{z^c} \int_0^z t^{c-1}[f(t)]^{\alpha+i\beta} \, dt]^{1/(\alpha+i\beta)}$$

is also Bazilevič of type (α,β) in U.

It is easy to see that the function f that arises from (4.1.2) when $h(z) \equiv 1$ must satisfy

$$(4.1.5) \qquad \text{Re}\{.1 + \frac{zf''(z)}{f'(z)} + (k-1) \frac{zf'(z)}{f(z)}\} > 0, \; z \in U,$$

with $k = \alpha+i\beta$, Re $k > 0$. Conversely, if $f \in H$, $f(z)f'(z)/z \neq$ ($z \in U$), and f satisfies (4.1.5) for Re $k > 0$, then the function f can be written in the form (4.1.2) with $h(z) \equiv 1$.

In [28], Eeningenburg et al. showed that if $f \in H$ with $f(z)f'(z)/z \neq 0$ for $z \in U$, then for $k = \alpha+i\beta$, $\alpha > 0$, β real, one has

$$(4.1.6) \quad \text{Re } \{1 + \frac{zf''(z)}{f'(z)} + (k-1) \frac{zf'(z)}{f(z)}\} > 0, \; z \in U,$$

implies $f \in S^{\lambda}(0)$,

where λ satisfies $\lambda = \arg(\alpha+i\beta)$, $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$.

Further in [57], Lewandowski et al. showed that if $f \in H$ for which $zf'(z)/f(z)$, $1 + (zf''(z)/f'(z))$ are nonvanishing in U and ν is a real number, then

$$(4.1.7) \quad \text{Re } \{(\frac{zf'(z)}{f(z)})^{1-\nu} (1 + \frac{zf''(z)}{f'(z)})^{\nu}\} > 0, \ z \in U \text{ implies } f \in S$$

and in [58], it was shown that

$$(4.1.8) \quad \text{Re } \{(\frac{zf'(z)}{f(z)})(1 + \frac{zf''(z)}{f'(z)})\} > 0, \ z \in U \text{ implies } f \in S^*$$

In [135], Singh showed that for $f \in H$ and $\mu = 1,2,\dots,$

$$(4.1.9) \quad \text{Re } \{f'(z) (\frac{f(z)}{z})^{\mu-1}\} > 0 \text{ implies } \text{Re}(\frac{f(z)}{z})^{\mu} > 0, \ z \in U$$

whereas Owa and Obradović [89] improved this result by showing that

$$(4.1.10) \quad \text{Re } \{f'(z)(\frac{f(z)}{z})^{\mu-1}\} > \beta$$

$$\text{implies } \text{Re } (\frac{f(z)}{z})^{\mu} > \frac{2\beta\mu+1}{2\mu+1} , \ z \in U$$

where $\mu > 0$ and $0 \le \beta < 1$. Simple examples can be constructed to show that the above results are not sharp.

We now introduce a class, the results about which would lead to sharpening and generalization of some of the above-mentioned results.

DEFINITION 4.1.1 :   A function f $\in$ H is said to be in B(μ,h) if

(4.1.11) $\dfrac{zf'(z)}{g^{\mu}(z)\ f^{1-\mu}(z)} \prec h(z),\ z \in U$

for some $\mu(\mu \geq 0)$, h a univalent function with h(0) = 1 and g $\in$ H.(In (4.1.11) powers are assumed to take principal values).

We denote by $B_1(\mu;h)$ the subclass of functions in B(μ;h) for which g(z) $\equiv$ z.   Thus B(0;h) $\equiv B_1$(0;h), is the class of functions f $\in$ H satisfying $\dfrac{zf'(z)}{f(z)} \prec h(z)$ for z $\in$ U, and B_1(1;h) is the subclass of H consisting of functions for which f'(z) $\prec$ h(z), z $\in$ U.   We also denote by B(μ,β) the subclass of B(μ;h) when h is taken to be h(z) = (1+(1-2β)z)/(1-z) and B_1(μ,β) is the subclass of B(μ,β) when g(z) $\equiv$ z and β < 1.

In this chapter, we propose to take up some application. of differential subordination to certain conformal mappings. These applications not only lead to improvement and sharpening of many of the earlier known results for the subclass of Bazilevič functions of type (α,β) but also give rise to a number of new results for the other subclasses as well.   This is accomplished by studying the wider class B(μ;h) introduced above.   In Section 4.2, we obtain some interesting results concerning B_1(μ,β) and give some sufficient conditions for a function to be close-to-convex (univalent), starlike and

convex respectively. In Section 4.3, an application of differential subordination to the investigation of certain integral transforms of the class $B(\mu;h)$ leads, perhaps for the first time, to sharp results in this direction. Further, the use of differential subordination has been made to prove sharp results for certain integral transform for the subclass $B_1(\mu,\beta)$. In Section 4.4, we improve the relation (4.1.6) and (4.1.7) by showing that the same conclusion holds under much weaker conditions on f. We also generalize and improve (4.1.8). Finally in addition to obtaining some results concerning the class $B_1(\mu,\beta)$, further sufficient conditions in terms of Schwarzian derivative for a function to be starlike or convex have also been obtained.

4.2 SOME SUFFICIENT CONDITIONS FOR A FUNCTION f TO BE CONVEX, STARLIKE, CLOSE-TO-CONVEX AND $(f(z)/z)^\mu \prec h(z)$:

We first state

THEOREM 4.2.1 : Let $f \in H$, $f \neq 0$ in $0 < |z| < 1$ h, be a convex univalent function in U with $h(0) = 1$, $\mu > 0$ and $\alpha \neq 0$ with Re $\alpha \geq 0$. If f satisfies

$$(4.2.1) \quad (1-\alpha)\,(\frac{f(z)}{z})^\mu + \alpha f'(z)(\frac{f(z)}{z})^{\mu-1} \prec h(z), \ z \in U$$

then

$$(4.2.2) \quad (\frac{f(z)}{z})^\mu \prec \frac{\mu}{\alpha}\, z^{-(\mu/\alpha)} \int_0^z t^{(\mu/\alpha)-1} h(t)dt, \ z \in U$$

and the result is sharp.

<u>Proof</u> :   Let h be a convex univalent function in U and h(0) = 1.   Consider

(4.2.3) $p(z) = (\frac{f(z)}{z})^{\mu}$ $(\mu > 0).$

Then p is analytic in U and p(0) = 1.   Differentiating both sides of (4.2.3) logarithmically, a simple computation yields

(4.2.4) $(1-\alpha)(\frac{f(z)}{z})^{\mu} + \alpha f'(z) (\frac{f(z)}{z})^{\mu-1} = p(z) + \frac{\alpha}{\mu} zp'(z).$

Therefore by Lemma 2.2.4, since f satisfies (4.2.1) and (4.2.4), we have

$$p(z) = (\frac{f(z)}{z})^{\mu} \prec \frac{\mu}{\alpha} z^{-(\mu/\alpha)} \int_{0}^{z} t^{(\mu/\alpha)-1} h(t)dt.$$

This proves (4.2.2) and the sharpness of the result follows from Lemma 2.2.4. Hence the theorem.

<u>REMARK 4.2.1</u> :   The case $\mu = 1$ of the above theorem leads to a result of Theorem 2.2.2.

Taking $\alpha = 1$ in Theorem 4.2.1 we obtain

<u>COROLLARY 4.2.1</u> :   <u>If</u> $f \in B_1(\mu;h)$ <u>and</u> $f \neq 0$ <u>in</u> $0 < |z| < 1$, <u>then</u>

$$(\frac{f(z)}{z})^{\mu} \prec \frac{\mu}{z^{\mu}} \int_{0}^{z} t^{\mu-1} h(t)dt \prec h(z), \ z \in U$$

<u>where</u> $\mu > 0$, <u>h being a convex univalent function in</u> U. <u>The result is sharp.</u>

For $h(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leqq \beta < 1)$, the above corollary improves the recent results of Owa and Obradovič [89] and

Singh [138].

Using the well-known sharp result of Suffridge [145], viz.,

(4.2.5) $zp'(z) \prec \varphi(z)$ implies $p(z) \prec \int_0^z t^{-1} \varphi(t)\, dt$

where $\varphi \in S^*$ and p, defined by $p(z) = z + \ldots$ analytic in U, we obtain

THEOREM 4.2.2 : Let $f \in H$, $f \neq 0$ in $0 < |z| < 1$ and $\varphi \in S^*$ then for μ real and $\mu > 0$,

(4.2.6) $f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - \left(\frac{f(z)}{z}\right)^{\mu} \prec \varphi(z)$, $z \in U$

        implies $\left(\frac{f(z)}{z}\right)^{\mu} - 1 \prec \mu^{-1} \int_0^z t^{-1} \varphi(t)\, dt$, $z \in U$.

The result is sharp.

Proof : If we set

(4.2.7) $p(z) = \frac{1}{\mu}\left[\left(\frac{f(z)}{z}\right)^{\mu} - 1\right]$, $(\mu > 0)$

then p is regular in U and $p(0) = 0$. A computation shows that

$$zp'(z) = f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - \left(\frac{f(z)}{z}\right)^{\mu}$$

holds for $z \in U$. Therefore (4.2.6) is equivalent to $zp'(z) \prec \varphi(z)$. Now the conclusion follows from (4.2.5) and (4.2.7). Hence the theorem.

Taking $\mu = 1$ in the above theorem, it follows that if $f \in H$ and $\varphi \in S^*$ satisfies

$$f'(z) - \frac{f(z)}{z} \prec \varphi(z), \ z \in U$$

then

$$\frac{f(z)}{z} - 1 \prec \int_0^z t^{-1} \varphi(t) \, dt, \ z \in U,$$

or if $f \in H$ and $\varphi \in S^*$ then,

$$zf''(z) \prec \varphi(z) \text{ implies } f'(z) - 1 \prec \int_0^z t^{-1} \varphi(t) dt, \ z \in U.$$

For example if $f \in H$, then for different choices of the starlike function $\varphi(z)$ in the above relation we obtain the following results.

COROLLARY 4.2.2 : Let $f \in H$ then

(a) $zf''(z) \prec ze^{kz}$ implies $f'(z) - 1 \prec \dfrac{e^{kz} - 1}{k}$, $z \in U$

for k real and $0 < k \leq 1$;

(b) $zf''(z) \prec -\dfrac{2z}{(1-z)^2}$ implies $f'(z) \prec \dfrac{1+z}{1-z}$, $z \in U$;

(c) $zf''(z) \prec z$ implies $f'(z) - 1 \prec z$, $z \in U$;

(d) $zf''(z) \prec \dfrac{z - (k/(k+1))z^2}{1-z}$

$\qquad\qquad$ implies $f'(z) - 1 \prec \dfrac{1}{k+1}[kz - \log(1-z)]$, $z \in U$,

$\qquad$ for all k, $|k - \frac{1}{8}| \leq \frac{3}{8}$;

(e) $zf''(z) \prec \dfrac{z}{1+z}$ implies $f'(z) - 1 \prec \log(1+z)$, $z \in U$.

THEOREM 4.2.3 : Let f $\in$ H and $\beta < 1$. If α, λ be complex numbers with Re $\alpha > 0$ and $|\lambda| \leq \frac{\text{Re } \alpha}{|\alpha|}$, then

$(4.2.8)$ Re $\{(1+\lambda z) [(1+\alpha\lambda z)f'(z)+\alpha(1+\lambda z)zf''(z)]\} > \beta$, $z \in U$
implies

$(4.2.9)$ Re $\{(1+\lambda z)f'(z)\} > \dfrac{2\beta + \text{Re } \alpha - |\alpha| \ |\lambda|}{2 \ + \text{Re } \alpha - |\alpha| \ |\lambda|}$, $z \in U$.

Proof :   Let

$(4.2.10)$ $\beta_1 = \dfrac{2\beta + \text{Re } \alpha - |\alpha| \ |\lambda|}{2 \ + \text{Re } \alpha - |\alpha| \ |\lambda|}$

and consider

$(4.2.11)$ $p(z) = (1-\beta_1)^{-1} [(1+\lambda z)f'(z) - \beta_1]$.

Then p is analytic in U and $p(0) = 1$. Again by setting

$r(z) = \alpha(1+\lambda z)$, (so that Re $(r(z)) >$ Re $\alpha - |\alpha\lambda| = \eta \geq 0$)

a simple computation shows that

$$(1+\lambda z)[(1+\alpha\lambda z)f'(z) + \alpha(1+\lambda z)zf''(z)]$$

$$= \beta_1 + (1-\beta_1) [p(z) + zp'(z) (r(z))]$$

$(4.2.12)$ $\equiv \Psi(p(z),zp'(z);z)$

where

$(4.2.13)$ $\Psi(r,s;z) = \beta_1 + (1-\beta_1) [r+s(r(z))]$.

Using $(4.2.8)$ and $(4.2.12)$, we obtain that for each $z \in U$,

$\{\Psi(p(z),zp'(z);z):z \in U\} \subset \Omega = \{w \in \mathbb{C} : \text{Re } w > \beta\}$.

Now for all real r_2 and $s_1 \leqq - \frac{1}{2} (1+r_2^2)$, we have for each $z \in U$,

$$\mathrm{Re}\ \{\Psi(ir_2,s_1;z)\} = \beta_1 + (1-\beta_1)s_1\ [\mathrm{Re}\ r(z)]$$

$$\leqq \beta_1 - \frac{(1-\beta_1)}{2}\ \eta$$

$$= \beta_1 - \frac{(1-\beta_1)}{2}\ (\mathrm{Re}\ \alpha - |\alpha|\ |\lambda|) \equiv \beta\ .$$

Hence for each $z \in U$, $\Psi(ir_2,s_1;z) \notin \Omega$. Thus by Lemma 2.4.1, $\mathrm{Re}\ p(z) > 0$ in U and hence by (4.2.11) we establish our claim.

REMARK 4.2.2 : It may be noted that the above theorem gives a sufficient condition for a function $f \in H$ to be close-to-convex(univalent)with respect to the starlike function $g(z) = z/(1+\lambda z)$, $z \in U$, when $-[\mathrm{Re}\ \alpha - |\alpha|\ |\lambda|]/2 \leqq \beta < 1$.

If we take α real and positive, $\beta = 0$ and set

$$(4.2.14) \quad v(z) = (1+\lambda z)\ [(\tfrac{1}{\alpha} + \lambda z)f'(z) + (1+\lambda z)zf''(z)],$$

the above theorem reduces for $|\lambda| \leqq 1$ to

$$(4.2.15) \quad \mathrm{Re}\{v(z)\} > 0 \text{ implies Re } \{(1+\lambda z)f'(z)\} > \frac{\alpha(1 - |\lambda|)}{2+\alpha(1-|\lambda|)}, z$$

Letting $\alpha \longrightarrow +\infty$, (4.2.15) is seen to be equivalent to

$$(4.2.16) \quad \mathrm{Re}\{w(z)\} \geqq 0 \text{ implies Re } \{(1+\lambda z)f'(z)\} \geqq 1 \ (|\lambda| \leqq 1)$$

where

$$w(z) = (1+\lambda z)\ [\lambda zf'(z) + (1+\lambda z)zf''(z)].$$

The relation (4.2.16) cannot be true for functions other than $f(z) = \lambda^{-1} \log (1+\lambda z)$.

In the following theorem we extend the result (4.2.16) as follows.

THEOREM 4.2.4 : Let $f \in H$, $\beta < 1$ and $|\lambda| \leq 1$. Then

$$\text{Re } \{(1+\lambda z)[\lambda z f'(z)+(1+\lambda z)z f''(z)] \; > \; \frac{-(1-\beta)}{2} \, (1-|\lambda|), \; z \in U$$

implies Re $\{(1+\lambda z)f'(z)\} > \beta$, $z \in U$.

Proof : Let

$$p(z) = (1-\beta)^{-1} [(1+\lambda z)f'(z) - \beta].$$

Then p is analytic in U and $p(0) = 1$. Again by setting $r(z) = 1+\lambda z$, (so that $\text{Re}(r(z)) > 1-|\lambda| = \eta \geq 0$), a simple computation shows that

$$(1+\lambda z)(\lambda z f'(z) + (1+\lambda z)z f''(z))$$

$$= (1-\beta)z p'(z) \, (r(z))$$

$$= \Psi(p(z), z p'(z); z),$$

where $\Psi(r, s; z) = (1-\beta)s \cdot (r(z))$. The remaining part of the proof follows on the similar lines as those of Theorem 4.2.3. This completes the proof of the theorem.

REMARK 4.2.3 : If we replace $z f'(z)$ by f in the above theorem we obtain that for $f \in H$, $\beta < 1$ and $|\lambda| \leq 1$,

$$(4.2.17) \quad \mathrm{Re}\ \{(1+\lambda z)[-\tfrac{f(z)}{z} + (1+\lambda z)f'(z)]\} > -\tfrac{(1-\beta)}{2}(1-|\lambda|), z \in U$$

$$\text{implies } \mathrm{Re}\ \{\frac{f(z)}{(z/(1+\lambda z))}\} > \beta,\ z \in U$$

$$\text{and } \mathrm{Re}\ \{\frac{zf'(z)}{z/(1+\lambda z)^2}\} > \frac{\beta(3-|\lambda|)-(1-|\lambda|)}{2},\ z \in U.$$

Theorem 4.2.4 gives another sufficient condition for a function $f \in H$ to be close-to-convex(univalent) in U for $0 \le \beta < 1$ whereas (4.2.17) gives a sufficient condition for a function $f \in H$ to be close-to-convex (univalent) in U for $(1-|\lambda|)/(3-|\lambda|) \le \beta < 1$.

We let $\{f,z\}$ denote the Schwarzian derivative

$$\{\frac{f''(z)}{f'(z)}\}' - \frac{1}{2}\ \{\frac{f''(z)}{f'(z)}\}^2,\quad f \in H.$$

Using Lemma 2.2.4 and (4.2.21) we obtain the following theorem that relates the Schwarzian derivative of f to the starlikeness and convexity (and univalency) of f and can be proved in a manner similar to that of Theorems 4.2.3 and 4.2.5. We illustrate this as follows.

THEOREM 4.2.5 : Let $f \in H$, and $f(z)f'(z)/z \neq 0$ for $z \in U$. Then for $\alpha \neq 0$, $\mathrm{Re}\ \alpha \ge 0$ and $h(z) = 1+h_1 z+\ldots$, a convex univalent function in U, we have

$$(4.2.16) \quad (1+\alpha)\frac{zf'(z)}{f(z)} + \alpha z^2[\{\int_0^z f,z\} + \frac{1}{2}(\frac{f'(z)}{f(z)})^2] \prec h(z)$$

$$\text{implies } \frac{zf'(z)}{f(z)} \prec \frac{1}{\alpha} z^{-(1/\alpha)}\int_0^z t^{(1/\alpha)-1}h(t)dt,\ z \in U,$$

and

$$(4.2.17) \quad 1+(1+\alpha)\frac{zf''(z)}{f'(z)}+\alpha z^2\left[\{f,z\}+\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right] \prec h(z)$$

$$\underline{implies}\ 1+\frac{zf''(z)}{f'(z)} \prec \frac{1}{\alpha}z^{-(1/\alpha)}\int_0^z t^{(1/\alpha)-1}h(t)dt,\ z \in U.$$

$\underline{Furthermore\ for}$ f $\in$ H $\underline{and}$ $\varphi \in S^*$, $\underline{we\ have}$,

$$(4.2.18) \quad \frac{zf'(z)}{f(z)}+z^2\left[\{\int_0^z f,z\}+\frac{1}{2}\left(\frac{f'(z)}{f(z)}\right)^2\right] \prec \varphi(z)$$

$$\underline{implies}\ \frac{zf'(z)}{f(z)}-1 \prec \int_0^z \frac{\varphi(t)}{t}\,dt,\ z \in U,$$

and

$$(4.2.19) \quad \frac{zf''(z)}{f'(z)}+z^2\left[\{f,z\}+\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right] \prec \varphi(z)$$

$$\underline{implies}\ \frac{zf''(z)}{f'(z)} \prec \int_0^z \frac{\varphi(t)}{t}\,dt,\ z \in U$$

$\underline{The\ results\ are\ sharp.}$

$\underline{Proof}$:   If we let

$$(4.2.20) \quad p(z) = \frac{zf'(z)}{f(z)}\ or\ \frac{zf''(z)}{f'(z)}+1$$

then as in the proof of Theorem 4.2.1, we see that (4.2.16) and (4.2.17) are equivalent to

$$p(z) + zp'(z).\alpha \prec h(z)\ ,\ z \in U$$

where h is convex univalent function in U. Now the conclusion follows from (4.2.20) and Lemma 2.2.4.

Similarly if one puts

$$p(z) = \frac{zf'(z)}{f(z)} \text{ or } \frac{zf''(z)}{f'(z)} + 1$$

then (4.2.18) and (4.2.19) are equivalent to writing

$$zp'(z) \prec \varphi(z), \ z \in U.$$

Now the conclusions follow from (4.2.5). This completes the proof of the theorem.

REMARK 4.2.4 : It may be noted that for different choices of convex functions h or starlike functions φ, we can obtain sufficient conditions for various subclasses of starlike and convex functions as obtained in Corollary 4.2.2, interms of Schwarzian derivative respectively.

4.3   INTEGRAL TRANSFORMS :

First we generalize and improve the result of Ruscheweyh [111] as follows:

THEOREM 4.3.1 : Let f $\in$ H and h be convex univalent function in U with h(O) = 1. Let μ be a real number with $\mu > 0$ and c be a complex number with Re $(\mu+c) > 0$ and suppose g $\in$ H satisfies the property that

$$(4.3.1) \qquad \mu \frac{zg'(z)}{g(z)} + c \prec Q_{\mu+c}(z), \quad z \in U.$$

Then for F(z)/z $\neq$ O in U,

$$(4.3.2) \quad \frac{zf'(z)}{g^\mu(z)f^{1-\mu}(z)} \prec h(z) \text{ implies } \frac{zF'(z)}{G^\mu(z) F^{1-\mu}(z)} \prec h(z), z \in U$$

where

(4.3.3) $\qquad F(z) = [\dfrac{\mu+c}{z^c} \int_0^z f^\mu(t)\, t^{c-1}\, dt]^{1/\mu}$,

(4.3.4) $\qquad G(z) = [\dfrac{\mu+c}{z^c} \int_0^z g^\mu(t)\, t^{c-1}\, dt]^{1/\mu}$

and $Q_{\mu+c}$ is the function defined as in Lemma 3.3.1.

Proof : By Lemma 3.3.1, (4.3.1) implies that the function G defined by (4.3.4) is analytic in U, $G(z)/z \neq 0$ and Re $\{\mu \dfrac{zG'(z)}{G(z)} + c\} > 0$ in U. If we set

(4.3.5) $\qquad p(z) = \dfrac{zF'(z)}{G^\mu(z)\, F^{1-\mu}(z)}$

then the standard arguments as used by Ruscheweyh [111] and others show that p is analytic in U and $p(0) = 1$. A simple calculation shows

(4.3.6) $\quad \dfrac{zf'(z)}{g^\mu(z)\, f^{1-\mu}(z)} = p(z) + zp'(z)\,(\lambda(z)),\ z \in U$

where $\lambda(z) = 1/(\mu \dfrac{zG'(z)}{G(z)} + c)$ and so Re $\lambda(z) > 0$ in U.

Since $f \in B(\mu;h)$, $h(z) = 1+h_1 z + \ldots$, is convex univalent in U, (4.3.6) gives that $p(z)+zp'(z)(\lambda(z)) \prec h(z)$, $z \in U$. It is clear that all the conditions of Lemma 3.2.1 are satisfied and so by that lemma we have $p(z) \prec h(z)$ in U. Hence by (4.3.5) the theorem follows.

Next, given F, the function f, given by (4.3.3), is written such that

(4.3.7) $f(z) = F(z) \{(c+\mu z F'(z)/F(z))/(c+\mu)\}^{1/\mu}$.

When μ tends to zero, the subordination relation (4.1.11) becomes $zf'(z)/f(z) \prec h(z)$, and at the same time the above relation (4.3.7) reduces to

(4.3.8) $f(z) = F(z) \exp \{c^{-1}(zF'(z)/F(z) - 1)\}$.

for $c \neq 0$. It follows from (4.3.8) that

(4.3.9) $F(z) = f(z) \exp \{-z^{-c} \int_{0}^{z} t^{c}(f'(t)/f(t) - t^{-1})\}dt$

for Re $c \geq 0$ and $c \neq 0$.

We can use Lemma 2.2.4 to improve and generalize the result of Yoshikawa and Yoshikai [151, Theorem 4] concerning the above transform (4.3.9) of λ-spiral-like functions.

THEOREM 4.3.2 :   Let $f \in H$, c be a complex number with Re $c \geq 0$, $c \neq 0$, h be a convex univalent function in U with h(0) = 1, and F be defined by (4.3.9). Then $f \in \mathcal{J}(0;h)$ i.e.,

(4.3.10) $\dfrac{zf'(z)}{f(z)} \prec h(z)$, $z \in U$

implies

(4.3.11) $\dfrac{zF'(z)}{F(z)} \prec cz^{-c} \int_{0}^{z} t^{c-1} h(t) \, dt$, $z \in U$.

The result is sharp.

Proof : When we put $p(z) = \frac{zF'(z)}{F(z)}$ we have from (4.3.8)

$$\frac{zf'(z)}{f(z)} = p(z) + c^{-1} zp'(z) .$$

Since $f \in H$ satisfies (4.3.10), the conclusion of the theorem follows from Lemma 2.2.4. Hence the theorem.

From (4.3.10) and (4.3.11) it follows that for $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$,

(4.3.12) $e^{i\lambda} \frac{zf'(z)}{f(z)} \prec e^{i\lambda} h(z)$

implies $e^{i\lambda} \frac{zF'(z)}{F(z)} \prec e^{i\lambda} [cz^{-c} \int_0^z t^{c-1}h(t)dt].$

By choosing

(4.3.13) $h(z) = \frac{1 + e^{-i\lambda} (2\rho \cos \lambda - e^{-i\lambda})z}{1+z}, \quad \rho < 1, -\pi/2 < \lambda < \pi/2$

(4.3.12) can be written as

(4.3.14) Re $\{e^{i\lambda} \frac{zf'(z)}{f(z)}\} > \rho \cos \lambda$

implies $e^{i\lambda} \frac{zF'(z)}{F(z)} \prec e^{i\lambda} [cz^{-c} \int_0^z t^{c-1} h(t)dt]$

where $h(z)$ is given by (4.3.13) and $F(z)$ is given by (4.3.9). The relation (4.3.14) is the best possible. The case $\rho = 0$ of (4.3.14) improves the result of Yoshikawa and Yoshikai [151, Theorem 4], who proved $f \in S^\lambda(0)$ implies $F \in S^\lambda(Re(1/c)/(2+Re(1/c)))$.

THEOREM 4.3.3 : <u>Let</u> μ <u>be a real number with</u> $\mu > 0$ <u>and</u> c <u>be a complex number with</u> Re $(\mu+c) > 0$. <u>Suppose that</u> f $\in$ H <u>and</u> h <u>be a convex univalent function in</u> U <u>with</u> h(O) = 1. <u>Then for</u> f $\in B_1(\mu;h)$ <u>and</u> F(z)/z $\neq$ O <u>in</u> U <u>we have</u>

$$(4.3.15) \qquad F'(z) \, (\frac{F(z)}{z})^{\mu-1} \prec \frac{\mu+c}{z^{\mu+c}} \int_0^z t^{\mu+c-1} \, h(t) \, dt$$

<u>where</u> F <u>is defined by</u> (4.3.3).

Proof : Set

$$(4.3.16) \qquad p(z) = F'(z) \, (\frac{F(z)}{z})^{\mu-1}.$$

Then p is analytic in U, with p(O) = 1.
A simple calculation shows that

$$(4.3.17) \qquad f'(z) \, (\frac{f(z)}{z})^{\mu-1} = p(z) + \frac{zp'(z)}{\mu+c} \, , \, z \in U.$$

By Lemma 2.2.4, since f $\in B_1(\mu;h)$ and it satisfies (4.3.17), we have

$$p(z) \prec \frac{\mu+c}{z^{\mu+c}} \int_0^z t^{\mu+c-1} \, h(t) dt, \, z \in U$$

and so by (4.3.16) we obtain our conclusion. Hence the theorem.

REMARK 4.3.1 : Taking μ = c = 1 we see that for f $\in$ H satisfying f'(z) $\prec$ h(z) in U for a convex univalent function h in U with h(O) =1 then the corresponding Libera transform $F(z) = \frac{2}{z} \int_0^z f(t) dt$ satisfies $F'(z) \prec \frac{2}{z^2} \int_0^z t \, h(t) dt$ and the result is the best possible. This extends an earlier

result of Libera [61] viz. (3.3.1) (with g(z) = z). For instance if f ∈ H and $F(z) = \frac{2}{z} \int_0^z f(t)dt$ then we have the sharp results

$$f'(z) \prec \frac{1+z}{1-z} \text{ implies } F'(z) \prec 1 + \frac{4}{z^2} [\frac{z^3}{3} + \frac{z^5}{5} + \ldots]$$

$$= -1 - \frac{4}{z} [1 + \frac{1}{z} \log(1-z)], z \in U;$$

and $f'(z) \prec 1+\lambda z$, $(0 \neq \lambda \in ¢)$ implies $F'(z) \prec 1 + \frac{2}{3} \lambda z$, $z \in U$.

4.4 SUFFICIENT CONDITIONS FOR A FUNCTION TO BE λ-SPIRAL-LIKE, STARLIKE AND CLOSE-TO-CONVEX

Recall that, Lewandowski et al. [57] showed that if f ∈ H for which zf'(z)/f(z), 1 + zf"(z)/f'(z) are non-vanishing in U and ν is a real number, then

(4.4.1) $Re \{(\frac{zf'(z)}{f(z)})^{1-\nu} (1 + \frac{zf''(z)}{f'(z)})^{\nu} \} > 0, z \in U,$ implies f ∈ S*

and in [58], it was shown that

(4.4.2) $Re \{(\frac{zf'(z)}{f(z)})(1 + \frac{zf''(z)}{f'(z)})\} > 0, z \in U$ implies f ∈ S*.

Further in [28], it was also proved that if f ∈ H and f(z)f'(z)/z ≠ 0 for z ∈ U, then

(4.4.3) $Re \{1 + \frac{zf''(z)}{f'(z)} + (\alpha+i\beta-1) \frac{zf'(z)}{f(z)}\} > 0, z \in U$

implies f ∈ $S^{\lambda}(0)$

where $\lambda = \arg(\alpha + i\beta)$, $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$.

The case when $\beta = 0$ and $\alpha = 1/\alpha'$ $(\alpha' > 0)$ is the well-known class of α'-convex functions considered by Mocanu [83] for $\alpha' > 0$ and others. In this section we improve all the above mentioned relations.

THEOREM 4.4.1 : Let $f \in H$, $zf'(z)/f(z)$ and $1+(zf''(z)/f'(z))$ be non-vanishing in U. Then for $\nu > 1/2$,

(4.4.4) $\quad (\frac{zf'(z)}{f(z)})^{1-\nu} (1 + \frac{zf''(z)}{f'(z)})^{\nu} \prec H_\nu(z)$ implies $f \in S^*$

where H_ν is the function that maps U onto the complex plane slit along the half-lines Re $w = 0$,

$|\text{Im } w| \geq \sqrt{\nu} \, [3\nu/(2\nu-1)]^{\nu-(1/2)}$.

Proof : If we set $p(z) = \frac{zf'(z)}{f(z)}$, then p is analytic in U and $p(0) = 1$. Therefore a simple calculation shows that fo. $\nu > 1/2$ (4.4.4) can be equivalently written as

(4.4.5) $\quad \Psi(p(z),zp'(z)) \equiv p(z)(1 + \frac{zp'(z)}{p^2(z)})^\nu \prec H_\nu(z)$, $z \in U$,

where

$$\Psi(r,s) = r(1 + \frac{s}{r^2})^\nu .$$

We shall show that Ψ satisfies the condition (2.4.28) of Lemma 2.4.1 with $a = 1$ and $k = 1$ there, i.e.,

(4.4.6) $\quad \Psi(ir_2,s_1) = ir_2(1 - \frac{s_1}{r_2^2})^\nu \notin \Omega \equiv H_\nu(U)$,

for all real r_2 and s_1 satisfying $s_1 \leqq -\frac{1}{2}(1+r_2^2)$.

If $r_2 > 0$, then we deduce that

$$(4.4.7) \qquad r_2(1 - \frac{s_1}{r_2^2})^\nu \geqq r_2(\frac{3}{2} + \frac{1}{2r_2^2})^\nu.$$

It is easy to show that the minimum value of the right hand member of the inequality (4.4.7) is $\sqrt{\nu}\,[3\nu/(2\nu-1)]^{\nu-(1/2)}$ when $\nu > 1/2$. Similarly if $r_2 < 0$ we deduce that

$$r_2(1 - \frac{s_1}{r_2^2})^\nu \leqq -\sqrt{\nu}\,[3\nu/(2\nu-1)]^{\nu-(1/2)}, \text{ for } \nu > 1/2,$$

and $r_2(1 - \frac{s_1}{r_2^2})^\nu \longrightarrow \pm\infty$ according as $r_2 \longrightarrow 0^+$ or 0^-.

Therefore (4.4.5) implies that the condition (4.4.6) holds and the conclusion of Theorem 4.4.1 follows from Lemma 2.4.1

This is an improvement over the result of Lewandowski et al. [57] for $\nu > 1/2$.

REMARK 4.4.1 : For the case ν real and $\nu \leqq 1/2$, we see that for $f \in H$, $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ non-vanishing in U, the relation

$$(\frac{zf'(z)}{f(z)})^{1-\nu}(1 + \frac{zf''(z)}{f'(z)})^\nu \prec \frac{1+z}{1-z}, \text{ implies } f \in S^*$$

holds and this can be observed from (4.4.6).

REMARK 4.4.2 : In particular for $f \in H$, $zf'(z)/f(z) \neq 0$ and $1+(zf''(z)/f'(z)) \neq 0$ in U, we have for $\nu > 1/2$,

(4.4.8) $\left| \text{Im} \left\{ \left(\dfrac{zf'(z)}{f(z)} \right)^{1-\nu} \left(1 + \dfrac{zf''(z)}{f'(z)} \right)^{\nu} \right\} \right|$

$$< \sqrt{\nu} \; \left[\frac{3\nu}{2\nu - 1} \right]^{\nu - (1/2)}, \quad z \in U$$

implies $f \in S^*$

and

(4.4.9) $\left| \left(\dfrac{zf'(z)}{f(z)} \right)^{1-\nu} \left(1 + \dfrac{zf''(z)}{f'(z)} \right)^{\nu} - 1 \right|$

$$< \left\{ 1 + \nu \left[\frac{3\nu}{2\nu - 1} \right]^{2\nu - 1} \right\}^{1/2}, \quad z \in U$$

implies $f \in S^*$.

For $\nu = 1$, (4.4.8) and (4.4.9) respectively reduce to the recent results of Mocanu [84], namely,

$$\left| \text{Im} \left(\frac{zf''(z)}{f'(z)} \right) \right| < \sqrt{3}, \text{ implies } f \in S^*$$

and

$$\left| \frac{zf''(z)}{f'(z)} \right| < 2 \text{ implies } f \in S^*.$$

THEOREM 4.4.2 : Let $f \in H$ and $f(z)f'(z)/z \neq 0$ for $z \in U$. If f satisfies

(4.4.10) $1 + \dfrac{zf''(z)}{f'(z)} + (k-1) \dfrac{zf'(z)}{f(z)} \prec Q_k(z), \quad z \in U$

for a complex number k with $\text{Re } k > 0$, then $f \in S^\lambda(0)$, where $\lambda = \arg k$, $-\dfrac{\pi}{2} < \lambda < \dfrac{\pi}{2}$ and $Q_k(z)$ is the function that maps U onto the complex plane slit along the half-lines $\text{Re } w = 0$, $|\text{Im } w| \geq \dfrac{1}{\text{Re } k} \left[|k| \{2 \text{ Re } k + 1\}^{1/2} - \text{Im } k \right]$.

Proof :   If we set

(4.4.11) $p(z) = k \dfrac{zf'(z)}{f(z)}$, $z \in U$

then p is analytic in U and Re p(O) = Re k > O.
By (4.4.11), (4.4.10) can be equivalently written as

$$\Psi(p(z), zp'(z)) \equiv p(z) + \frac{zp'(z)}{p(z)} \not\subset Q_k(z), \ z \in U$$

where $\Psi(r, s) = r + \dfrac{s}{r}$.

As in Theorem 4.4.1, it can be easily verified that Ψ
satisfies the condition (2.4.28) of Lemma 2.4.1, i.e.,

$$i\left(r_2 - \frac{s_1}{r_2}\right) = \Psi(ir_2, s_1) \notin Q_k(U) \equiv \mathbf{O}$$

for all real r_2 and s_1 satisfying $s_1 \leq -\dfrac{1}{2\text{Re } k}\left[|k|^2 - 2r_2 \text{ Im } k + r_2^2\right]$

Now the conclusion of Theorem 4.4.2 follows from Lemma 2.4.1.
Hence the theorem.

REMARK 4.4.3 :   The above theorem shows that the same
conclusion holds under much weaker condition on f than (4.4.3).
This is an improvement of a result of Eeningenburg et al.
[28, Theorem 1].

In particular from the above theorem we easily obtain
that if $f \in H$ and $f(z)f'(z)/z \neq O$ in U then

(4.4.12) $\left| 1 + \dfrac{zf''(z)}{f'(z)} + (k-1)\dfrac{zf'(z)}{f(z)} - 1 \right| < [1 + c^2]^{1/2}$, $z \in U$

implies $f \in S^\lambda(O)$,

and

$$(4.4.13) \quad \left| \text{Im} \left\{ 1 + \frac{zf''(z)}{f'(z)} + (k-1) \frac{zf'(z)}{f(z)} \right\} \right| < C, \ z \in U$$

$$\text{implies } f \in S^{\lambda}(O),$$

where λ is given by $k = |k| \ e^{i\lambda}$, $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$

and $C = \frac{1}{\text{Re } (k)} \left[|k| \ \sqrt{1+2 \text{ Re } k} - \text{Im } k \right]$.

The equation (4.4.12) and (4.4.13) give new sufficient conditions for a function f to be λ-spiral-like in U.

REMARK 4.4.4 : Considering $k = 1/\alpha > O$, it follows from Theorem 4.4.2, that for $f \in H$, $f'(z)f(z)/z \neq O$ in U and $\alpha > O$, we have

$$(4.4.14) \quad 1 + \frac{zf''(z)}{f'(z)} + \left(\frac{1}{\alpha} - 1 \right) \frac{zf'(z)}{f(z)} \prec Q_{\frac{1}{\alpha}}(z) = \frac{1}{\alpha}\left(\frac{1+z}{1-z}\right) + \frac{2z}{1-z^2}$$

$$\text{implies } f \in S^{*}.$$

This was recently obtained in [84, Corollary 2.1].
Letting $\alpha \longrightarrow +\infty$, (4.4.14) yields

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2z}{1-z^2} \text{ implies } f \in S^{*}.$$

In the following theorem we obtain some more results of this type.

THEOREM 4.4.3 : Let $f \in H$ and $f(z)f'(z)/z \neq O$ in U. Then

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{4k\ e^{i\alpha}z}{1-z^2}, \ z \in U$$

$$\underline{implies} \ \frac{zf'(z)}{f(z)} \prec (\frac{1+z}{1-z})^{2ke^{i\alpha}}$$

<u>where</u> $\quad 0 < k \leq \cos \alpha \ , \ |\alpha| < \frac{\pi}{2}$;

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{z}{1+z} \ \underline{implies} \ \frac{zf'(z)}{f(z)} \prec 1+z, \ z \in U;$$

For $0 < \beta \leq 1$

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec \frac{2\beta z}{1-z} \ \underline{implies} \ \frac{zf'(z)}{f(z)} \prec (1-z)^{-2\beta} \ , \ z \in U.$$

<u>Proof</u> : Let $p(z) = \frac{zf'(z)}{f(z)}$.

Then p is analytic in U, $p(0) = 1$ and

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} \ .$$

Now using Theorem 3 of [76], it is easy to see that

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)} \ implies \ p(z) \prec q(z)$$

whenever $q(z)$ is univalent and $zq'(z)/q(z)$ is starlike in U. The conclusion of the theorem now follows by choosing $q(z)$ respectively as $4ke^{i\alpha}z/(1-z^2)$ $(0 < k \leq \cos \alpha, \ |\alpha| < \pi/2)$, $1+z$ and $(1-z)^{-2\beta}$, $0 < \beta \leq 1$

In [138, Theorem 4] Singh and Singh proved that if $f \in H$ satisfies

$$(4.4.15) \quad |\alpha(\frac{zf'(z)}{f(z)} - 1) + (1-\alpha) \frac{z^2f''(z)}{f(z)})| < 1, \ z \in U,$$

for $0 \leq \alpha < 1$, then $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$ in U. In the following theorem, we improve this result by showing that the same conclusion holds under weaker condition on f than (4.4.15).

THEOREM 4.4.4 : If $f \in H$ and $f(z)f'(z)/z \neq 0$ in U, satisfy

$$(4.4.16) \quad \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + (1-\alpha)\frac{z^2 f''(z)}{f(z)} \prec (2-\alpha)z + (1-\alpha)z^2, \quad z \in U$$

for $0 \leq \alpha \leq 1$, then $\frac{zf'(z)}{f(z)} \prec 1+z$ in U. The function f given by $f(z) = z\,e^z$ shows that the result is sharp.

Proof : Set $0 \leq \alpha < 1$. If we let

$$(4.4.17) \quad p(z) = \frac{zf'(z)}{f(z)} - 1 \,,$$

then (4.4.16) is equivalent to

$$(4.4.18) \quad \frac{p(z)(1+(1-\alpha)p(z))}{1-\alpha} + zp'(z) \prec \frac{(2-\alpha)}{1-\alpha}z + z^2, \quad z \in U.$$

We claim that $p(z) \prec z$. For this if we set

$$\Theta(z) = \frac{1}{1-\alpha}\left[z(1 + (1-\alpha)z)\right]$$

then Θ is analytic in a domain $D \supset U$ and $Re(\Theta'(z)) > -1$ for $z \in U$ and $0 \leq \alpha < 1$. Then by using a result of Miller and Mocanu [76, Corollary 3.1] we obtain that (4.4.18) implies $p(z) \prec z$. This by (4.4.17) gives the required implication.

REMARK 4.4.5 : Taking $\alpha = 0$ in the above theorem it follows that for $f \in H$,

$$\frac{z^2 f''(z)}{f(z)} \prec (1+z)^2 - 1 \text{ implies } \left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \ z \in U.$$

THEOREM 4.4.5 : Let $f \in H$ for which $zf'(z)/f(z)$ be nonvanishing in U, then for every odd integer $n \geq 1$,

$$(4.4.19) \quad \left(kz \frac{f'(z)}{f(z)}\right)^n \left(1 + \frac{zf''(z)}{f'(z)} + (k-1) \frac{zf'(z)}{f(z)}\right)^n \prec G(z), \ z \in U$$

implies $f \in S^\lambda(0)$,

where k, a complex number with Re k > 0 and G is the function that maps U onto the complex plane slit along the negative real axis from

$$- \left[\frac{1}{2 \ Re \ k} \ \{|k|^2 - (Im \ k)^2 (1 + 2 \ Re \ k)^{-1}\}\right]^n \text{ to infinity}$$

and λ satisfies $k = |k| \ e^{i\lambda}$, $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$.

The above theorem can be proved on the same lines as those of Theorem 4.4.1 and Theorem 4.4.2 by taking $p(z) = k \frac{zf'(z)}{f(z)}$ and so writing (4.4.19) as

$$p^n(z) \left(p(z) + \frac{zp'(z)}{p(z)}\right)^n \equiv (p^2(z) + zp'(z))^n \prec G(z), \ z \in U$$

where G is as defined in Theorem 4.4.5.

Taking n = 1 and k = 1 in the above theorem we have the following corollary which improves (4.4.2) of Lewandowski et al. [58].

COROLLARY 4.4.1 :　Let f ∈ H and f(z)f'(z)/z ≠ O in U.　If f satisfies

$$\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^2 + \frac{2z}{(1-z)^2} \equiv G_1(z), \quad z \in U$$

then f ∈ S*, where G_1 is the function that maps U onto the entire plane minus the part of the negative real axis from $-\frac{1}{2}$ to infinity.

In particular the above subordination relation gives

$$\mathrm{Re} \left\{\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2} , \quad z \in U \text{ implies } f \in S*.$$

THEOREM 4.4.6 :　If f ∈ H and f(z) ≠ O in O < |z| < 1 satisfy

(4.4.21)　$\mathrm{Re} \left\{\left(\frac{f(z)}{z}\right)^{2\alpha-1} f'(z)\right\} > \frac{\beta((2\alpha+1)\beta-1)}{2\alpha} \equiv \beta_1(\alpha), \quad z \in U$

for some α > O and $\frac{1}{2(2\alpha+1)} \leqq \beta < 1$, then $\mathrm{Re} \left(\frac{f(z)}{z}\right)^\alpha > \beta$ in U.

Proof :　Let $\beta_1(\alpha) = \beta((2\alpha+1)\beta-1)/2\alpha$ and consider

(4.4.22)　$p(z) = (1-\beta)^{-1} \left[\left(\frac{f(z)}{z}\right)^\alpha - \beta\right].$

Then p is analytic in U and p(O) = 1.　A simple computation shows that

$$\left(\frac{f(z)}{z}\right)^\alpha \frac{f'(z)}{\left(\frac{f(z)}{z}\right)^{1-\alpha}} = \Psi(p(z), zp'(z))$$

where

(4.4.23)　$\Psi(r,s) = (\beta + (1-\beta)r)^2 + (\beta + (1-\beta)r) \frac{(1-\beta)}{\alpha} s,$

with $r = p(z)$ and $s = zp'(z)$.

Using (4.4.21) and (4.4.23), we obtain

$$\{\Psi(p(z),zp'(z)) : z \in U\} \subset \Omega = \{w \in \mathbb{C} : \text{Re } w > \beta_1(\alpha)\}.$$

Now for all real r_2 and s_1 satisfying $s_1 \leq -\dfrac{(1+r_2^2)}{2}$ we have

$$\text{Re } \{\Psi(ir_2,s_1)\} \leq \beta^2-(1-\beta)^2\, r_2^2 - \frac{(1-\beta)\beta}{2\alpha}\,(1+r_2^2)$$

$$\leq \beta^2 - \frac{\beta(1-\beta)}{2\alpha} \equiv \beta_1(\alpha)$$

i.e. $\Psi(ir_2,s_1) \notin \Omega$.

Thus by Lemma 2.4.1, Re $p(z) > 0$ in U and hence from (4.4.22) we obtain Re $(\frac{f(z)}{z})^\alpha > \beta$ in U.

REMARK 4.4.6 :   Taking $\beta = 1/(2\alpha+1)$ we obtain that for $f \in H$

(4.4.24)   Re $\{(\frac{f(z)}{z})^{2\alpha-1} f'(z)\} > 0$

$$\text{implies Re } (\frac{f(z)}{z})^\alpha > \frac{1}{2\alpha+1} , \; z \in U,$$

and taking $\beta = 1/2(2\alpha+1)$ we get

(4.4.25)   Re $\{(\frac{f(z)}{z})^{2\alpha-1} f'(z)\} > \frac{-1}{8\alpha(2\alpha+1)}$

$$\text{implies Re}(\frac{f(z)}{z})^\alpha > \frac{1}{2(2\alpha+1)} , \; z \in U.$$

(4.4.24) and (4.4.25) give that, for $f \in H$

$$\text{Re } \{\frac{f(z)}{z} f'(z)\} > 0 \text{ implies Re}\{\frac{f(z)}{z}\} > \frac{1}{3} , \; z \in U,$$

or   Re $\{f'(zf''+f')\} > 0$ implies Re $f'(z) > \frac{1}{3}$, $z \in U$

and

Re $\{\frac{f(z)}{z} f'(z)\} > -\frac{1}{24}$ implies Re $\frac{f(z)}{z} > \frac{1}{6}$, $z \in U$

or   Re $\{f'(zf''(z)+f')\} > -\frac{1}{24}$ implies Re $f'(z) > \frac{1}{6}$, $z \in U$.

The following theorem, which relates the Schwarzian derivative $\{f,z\} = (\frac{f''(z)}{f'(z)})' - \frac{1}{2} (\frac{f''(z)}{f'(z)})^2$ to starlikeness of order β and convexity of order $\tilde{\beta}$ of f, can be proved in a manner similar to that of Theorem 4.4.6. So we omit its proof.

THEOREM 4.4.7 : <u>For</u> f $\in$ H, $\frac{1}{6} \leq \beta < 1$ <u>we</u> <u>have</u>

$$\text{Re}\left[(1 + \frac{zf''(z)}{f'(z)})(z^2 \{f,z\} + \frac{1}{2}(1 + \frac{zf''(z)}{f'(z)})^2 + \frac{1}{2}(1+2 \frac{zf''(z)}{f'(z)})) \right] > \frac{1}{2}\beta(3\beta-1)$$

<u>implies</u> Re $\{1 + \frac{zf''(z)}{f'(z)}\} > \beta$, $z \in U$

<u>and</u>

$$\text{Re}\left[\frac{zf'(z)}{f(z)}(z^2 \{\int_0^z f,z\} + \frac{1}{2}(\frac{zf'(z)}{f(z)})^2 + 2 \frac{zf'(z)}{f(z)}) \right] > \frac{\beta(3\beta-1)}{2} , z \in U$$

<u>implies</u> Re $\{\frac{zf'(z)}{f(z)}\} > \beta$, $z \in U$.

For the proof of next two theorems we need the following lemma.

LEMMA 4.4.1 :   <u>Let</u> Ω <u>be</u> <u>a</u> <u>set</u> <u>in</u> <u>the</u> <u>complex</u> <u>plane</u> $\mathbb{C}$, $r = r_1+ir_2$, $s_1 = s_1+is_2$ <u>and</u> $t = t_1+it_2$. <u>Suppose</u> <u>that</u> <u>the</u> <u>function</u> $\Psi : \mathbb{C}^3 \longrightarrow \mathbb{C}$ <u>satisfies</u> <u>the</u> <u>condition</u>

$$\Psi(ir_2,s_1,t_1+it_2) \notin \Omega,$$

for all real r_2, s_1, t_1 satisfying $s_1 \leqq - \dfrac{k(1+r_2^2)}{2}$, $s_1 + t_1 \leqq 0$.

If p defined by $p(z) = 1 + p_k z^k + \ldots$, is analytic in U and $\Psi(p(z), zp'(z), z^2 p''(z)) \in \Omega$ when $z \in U$, then Re $p(z) > 0$ in U.

    More general form of this may be found in [75].

THEOREM 4.4.8 : Let $f \in H$ and α be a real number satisfying $\alpha \geq 0$ and $\frac{1}{2} \leq \beta < 1$. Then

$$\text{Re } \{\frac{zf'(z)}{f(z)} + 2\alpha(\frac{f(z)}{z} - f'(z) + zf''(z))\} > \frac{3\beta-1}{2\beta} \ , \ (z \in U)$$

$$\text{implies Re } \frac{f(z)}{z} > \beta, \ (z \in U).$$

Proof : Let $p(z) = (1-\beta)^{-1} (\frac{f(z)}{z} - \beta)$, $\frac{1}{2} \leqq \beta < 1$, $z \in U$.

Then p given by $p(z) = 1 + p_1 z + \ldots$, is analytic in U. It can be easily seen that

$$\frac{zf'(z)}{f(z)} + 2\alpha(\frac{f(z)}{z} - f'(z) + zf''(z)) = \Psi(p(z), zp'(z), z^2 p''(z))$$

where

$$\Psi(r,s,t) = 1 + \frac{(1-\beta)s}{\beta + (1-\beta)r} + 2\alpha(1-\beta)(s+t)$$

with $r = p(z)$, $s = zp'(z)$ and $t = z^2 p''(z)$.

Now by using Lemma 4.4.1 with $\Omega = \{w \in \mathbb{C} : \text{Re } w > \frac{3\beta-1}{2\beta}\}$, we can easily obtain the result. Hence the theorem.

    Taking $\alpha = 0$ and $\rho = \frac{1}{2}(3 - 1/\beta)$ in the above theorem we obtain for $f \in H$

$$\text{Re}\{\frac{zf'(z)}{f(z)}\} > \rho \ (\frac{1}{2} \leqq \rho < 1) \text{ implies Re } \frac{f(z)}{z} > \frac{1}{3-2\rho} \ , \ z \in U.$$

The case $\rho = 1/2$ was independently obtained by Marx [68] and Strohhäcker [143].

Finally we prove

THEOREM 4.4.9 : Let $f \in H$ with $f(z)f'(z)/z \neq 0$, then

(i) for $0 < \beta < 1$ and α real satisfying $\alpha > 1/2$

$$(4.4.26) \quad \text{Re } \left\{\frac{zf'(z)}{f(z)}\left((\alpha-1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2} + \frac{\beta(2\alpha\beta-1)}{2} \ ,$$

$$z \in U$$

implies $f \in S^*(\beta)$,

(ii) for a complex number α with $\text{Re } \alpha > 1/2$,

$$(4.4.27) \quad \text{Re } \left\{\frac{zf'(z)}{f(z)}\left((\alpha-1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2} \ , \ z \in U$$

implies $f \in S^*$,

and (iii)

$$(4.4.28) \quad \text{Re } \left\{\frac{zf'(z)}{f(z)}\left(z^2\{f,z\} + 1 + \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{1}{2} \ , \ z \in U$$

implies $f \in S^*$.

Proof : Set

$$p(z) = (1-\beta)^{-1} \left[\frac{zf'(z)}{f(z)} - \beta\right] \ , \ z \in U.$$

Then p is analytic in U, $p(0) = 1$ and

$$\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} + (\alpha-1)\frac{zf'(z)}{f(z)}\right) = \Psi(p(z), zp'(z)) \ ,$$

where

$$\Psi(r,s) = \alpha((1-\beta)r + \beta)^2 - (\beta + (1-\beta)r) + (1-\beta)s$$

with $r = p(z), \ s = zp'(z)$.

Now using arguments similar to those used to prove the earlier theorem, one can easily obtain (4.4.26) and (4.4.27).
For the proof of (4.4.28) we set

$$p(z) = \frac{zf'(z)}{f(z)} , \ z \in U.$$

Then p is analytic in U and $p(0) = 1$, and a simple calculation yields

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)} ,$$

and

$$z^2 \ \{f,z\} = \frac{zp'(z) + z^2 p''(z)}{p(z)} - \frac{1}{2} \left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{(1-p^2(z))}{2}.$$

Therefore

$$\frac{zf'(z)}{f(z)} \ (z^2\{f,z\} + 1 + \frac{zf''(z)}{f'(z)}) = \Psi(p(z),p'(z),z^2 p''(z))$$

where

$$\Psi(r,s,t) = r^2 + 2s + t - \frac{1}{2} \frac{s^2}{r} + \frac{r-r^3}{2} .$$

If we take $\Omega = \{w \in \mathbb{C} : \text{Re } w > -\frac{1}{2}\}$, one can easily obtain (4.4.28) from Lemma 4.4.1. Hence the theorem.

CERTAIN SUBCLASSES OF FUNCTIONS RELATED TO N-SYMMETRIC POINTS

5.1 INTRODUCTION :

A function f in H is said to be starlike with respect to symmetric points [120] if for every r close to 1, r < 1 and every z_o on $|z| = r$, the angular velocity of f(z) about the point $f(-z_o)$ is positive at $z = z_o$ as z traverses the circle $|z| = r$ in the positive direction, i.e.,

$$(5.1.1) \quad \text{Re } \left\{\frac{zf'(z)}{f(z)-f(-z_o)}\right\} > 0, \text{ for } z = z_o, \ |z| = r.$$

Analytically, it has been characterized [120] that, a function f in H is univalent and starlike with respect to symmetric points if and only if,

$$(5.1.2) \quad \text{Re } \left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0, \ z \in U.$$

Motivated by (5.1.1), it is natural to extend the definition of starlikeness with respect to N-symmetric points as follows.

For a positive integer N, let $\varepsilon = \exp(2\pi i/N)$ denote the Nth root of unity. For $f \in H$, let

$$(5.1.3) \quad M_{f,N}(z) = \left[\sum_{j=1}^{N-1} \varepsilon^{-j} f(\varepsilon^j z)\right] / \left(\sum_{j=1}^{N-1} \varepsilon^{-j}\right)$$

be its N-weighted mean function. A function f in H is said to belong to the class S_N^* of functions starlike with respect to N-symmetric points if for every r close to 1, r < 1, the angular velocity of f(z) about the point $M_{f,N}(z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction, i.e.,

$$(5.1.4) \quad \text{Re } \left\{ \frac{z f'(z)}{f(z) - M_{f,N}(z_0)} \right\} > 0, \text{ for } z = z_0, \ |z_0| = r.$$

It is clear that for N = 2, (5.1.4) reduces to (5.1.1).

Analytically [134], a function f in H is univalent and starlike with respect to N-symmetric points, or briefly N-starlike, if and only if,

$$(5.1.5) \quad \text{Re } \left\{ \frac{z f'(z)}{f_N(z)} \right\} > 0, \ z \in U$$

where

$$(5.1.6) \quad f_N(z) = \frac{1}{N} \left[f(z) - M_{f,N}(z) \right] = \frac{1}{N} \sum_{j=0}^{N-1} \varepsilon^{-j} f(\varepsilon^j z)$$

$$\equiv z + \sum_{m=1}^{\infty} a_{mN+1} z^{mN+1}.$$

Here in (5.1.6), a use has been made of the fact that

$$(5.1.7) \quad 1 + \varepsilon^k + \varepsilon^{2k} + \ldots + \varepsilon^{(N-1)k} = \begin{cases} N \text{ if k is multiple of N,} \\ 0 \quad \text{otherwise.} \end{cases}$$

The definitions of order and other subclasses related to N-starlike functions can be extended in a natural manner.

Thus we have the following definitions.

DEFINITION 5.1.1 : A function f in H is said to belong to the class $S_N^*(A,B)$ $(-1 \leq B < A \leq 1)$ if

$$(5.1.8) \qquad \frac{zf'(z)}{f_N(z)} \prec \frac{1+Az}{1+Bz} , \quad z \in U$$

where f_N is defined by (5.1.6).

We denote by $S_N^*(1-2\beta,-1) \equiv S_N^*(\beta)$ $(0 \leq \beta < 1)$ and $S_N^*(0) \equiv S_N^*$ and call $S_N^*(\beta)$ to be the class of functions starlike with respect to N-symmetric points of order β or N-starlike of order β $(0 \leq \beta < 1)$.

DEFINITION 5.1.2 : A function f in H is said to belong to the class $K_N(A,B)$ $(-1 \leq B < A \leq 1)$ if

$$(5.1.9) \qquad \frac{(zf'(z))'}{f_N'(z)} \prec \frac{1+Az}{1+Bz} , \quad z \in U.$$

We denote by $K_N(1-2\beta,-1) \equiv K_N(\beta)$ $(0 \leq \beta < 1)$ and $K_N(0) \equiv K_N$ and call $K_N(\beta)$ to be the class of functions convex with respect to N-symmetric points of order β or N-convex of order β $(0 \leq \beta < 1)$.

DEFINITION 5.1.3 : A function f in H is said to belong to the class $C_N(A,B)$ $(-1 \leq B < A \leq 1)$ if there exists g in S_N^* satisfying

$$(5.1.10) \qquad \frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz} , \quad z \in U.$$

We denote by $C_N(1-2\beta,-1) \equiv C_N(\beta)$ $(0 \leq \beta < 1)$ and $C_N(0) \equiv C_N$. We call a function in $C_N(\beta)$ as functions close-to-convex

with respect to N-symmetric points of order β or N-close-to-convex of order β $(0 \leq \beta < 1)$.

It may be noted that for $N = 1$, we have $f_N = f$ and the conditions (5.1.8), (5.1.9) and (5.1.10) show that $S_1^*(A,B) \equiv S^*(A,B)$, $K_1(A,B) \equiv K(A,B)$ and $C_1(A,B) \equiv C(A,B)$. Thus the classes $S_N^*(A,B)$, $K_N(A,B)$ and $C_N(A,B)$ are the generalizations of Sakaguchi's concept of functions starlike with respect to symmetric points [120]. Recently the classes S_N^*, K_N and C_N have been studied by Chand and Singh [23], Singh and Tygel [134], Singh and Singh [133] whereas the classes $S_N^*(\beta)$, $K_N(\beta)$ and $C_N(\beta)$ have been studied by Reddy [104].

In this chapter we first obtain, in Section 5.2, structural formulae for the classes $S_N^*(A,B)$, $K_N(A,B)$ and $C_N(A,B)$. In Section 5.3, using the convolution techniques we obtain necessary and sufficient condition for a function to be in $S_N^*(A,B)$ (or $K_N(A,B)$). In Section 5.4, we demonstrate the validity of Polya-Schoenberg conjecture for these classes by showing them to be closed under convolution with convex functions, which in turn further leads to results about de la Vallée poussin means and partial sums of functions in these classes. In Section 5.5 we obtain a result concerning neighbourhoods of analytic functions in these classes.

5.2 STRUCTURAL FORMULAE :

We first derive the following structural formulae for functions in the classes $S_N^*(A,B)$, $K_N(A,B)$ and $C_N(A,B)$ respectively.

THEOREM 5.2.1 : A function f belongs to the class $S_N^*(A,B)$ if and only if there exists a function p $\in$ P(A,B) such that

$$(5.2.1) \quad f(z) = \int_o^z p(t)[\exp\{\int_o^t \frac{1}{Nu}(\sum_{j=0}^{N-1} p(\varepsilon^j u) - N)du\}]\, dt$$

where $\varepsilon = \exp(2\pi i/N)$.

Proof : We first prove the necessity of (5.2.1). Suppose f $\in$ $S_N^*(A,B)$. Then by definition it follows that

$$(5.2.2) \qquad \frac{zf'(z)}{f_N(z)} = p(z)$$

where p $\in$ P(A,B). Replacing z by $\varepsilon^j z$ in (5.2.2) we obtain

$$(5.2.3) \qquad \frac{zf'(\varepsilon^j z)}{f_N(z)} = p(\varepsilon^j z) \ .$$

From (5.2.2) and (5.2.3),

$$(5.2.4) \qquad f'(\varepsilon^j z) = p(\varepsilon^j z)\, \frac{f'(z)}{p(z)}.$$

On the other hand, from (5.2.2),

$$(5.2.5) \qquad f_N(z) = \frac{zf'(z)}{p(z)} \ .$$

Differentiation of (5.2.5) on both sides yields

$$f_N'(z) = -p'(z)[p(z)]^{-2}zf'(z)+[zf''(z)+f'(z)][p(z)]^{-1}.$$

From (5.2.4),

$$(5.2.6) \qquad f_N'(z) = N^{-1} \sum_{j=0}^{N-1} f'(\varepsilon^j z) = \frac{f'(z)}{p(z)} [N^{-1} \sum_{j=0}^{N-1} p(\varepsilon^j z)].$$

Comparing (5.2.5) and (5.2.6) we have,

$$\frac{f''(z)}{f'(z)} = \frac{p'(z)}{p(z)} + \frac{1}{Nz} [\sum_{j=0}^{N-1} p(\varepsilon^j z) - N].$$

This gives after a repeated integration the structural formulae (5.2.1).

<u>Sufficiency</u> : Suppose the formula (5.2.1) holds with $p \in P(A,B)$. The function f is obviously analytic in U such that $f(0) = f'(0)-1 = 0$. We first verify by differentiation the identity

$$z \exp \{\int_0^t \frac{1}{Nt}(\sum_{j=0}^{N-1} p(\varepsilon^j t) - N)du\}$$

$$= \int_0^z \frac{1}{N} \sum_{j=0}^{N-1} p(\varepsilon^j t)[\exp \{\int_0^z \frac{1}{Nu}(\sum_{j=0}^{N-1} p(\varepsilon^j u)-N)du\}]\ dt$$

where $p \in P(A,B)$. Moreover, by (5.2.1)

$$(5.2.8) \qquad f'(z) = p(z) \exp \{\int_0^z \frac{1}{Nu}(\sum_{j=0}^{N-1} p(\varepsilon^j u)-N)du\}$$

which shows that $f' \neq 0$ in U. From (5.2.1), it can be easily seen by using the change of variables and the fact that ε is the Nth root of unity, that

$$(5.2.9) \quad f_N(z) = \int_0^z \frac{1}{N} \sum_{j=0}^{N-1} p(\varepsilon^j s) [\exp\{\int_0^s \frac{1}{N\eta} (\sum_{j=0}^{N-1} p(\varepsilon^j \eta) - N) d\eta\}] ds.$$

Using (5.2.7), (5.2.8) and (5.2.9) we finally obtain

$$f_N(z) = zf'(z)/p(z).$$

This proves the sufficiency of (5.2.1).

REMARK 5.2.1(i) : Using the fact that $f \in K_N(A,B)$ if and only if $zf' \in S_N^*(A,B)$, one can easily derive from the above theorem, the structural formula for a function in $K_N(A,B)$.

(ii) Taking $A = 1-2\beta$ ($0 \leq \beta < 1$) and $B = -1$ in Theorem 5.2.1 we obtain the structural formula for the class $S_N^*(\beta)$.

(iii) Taking $A = 1$, $B = -1$ and $N = 2$ in Theorem 5.2.1 we obtain the corresponding structural formula obtained by Stankiewicz [142] for functions starlike with respect to symmetric points.

(iv) Taking $N = 1$ and using the identity (5.2.7) we get the corresponding structural formula for functions in $S^*(A,B)$.

THEOREM 5.2.2 :   The function f belongs to the class $C_N(A,B)$ with respect to $g \in S_N^*$ if and only if there exist two functions p, q, p $\in$ P(A,B), q $\in$ P such that

$$(5.2.10) \quad f(z) = \int_0^z p(t)[\exp\{\int_0^t \frac{1}{Nu}(\sum_{j=0}^{N-1} q(\epsilon^j u) - N)du\}]\, dt$$

$$(5.2.11) \quad g(z) = \int_0^z q(t)[\exp\{\int_0^t \frac{1}{Nu}(\sum_{j=0}^{N-1} q(\epsilon^j u) - N)du\}]\, dt$$

<u>where</u> $\epsilon = \exp(2\pi i/N)$.

<u>Proof</u> : To prove the necessary part, suppose that f is in $C_N^*(A,B)$ with respect to $g \in S_N^*$. Then by definition it follows that

$$(5.2.12) \qquad \frac{zf'(z)}{g_N(z)} = p(z), \quad q(z) = \frac{zg'(z)}{g_N(z)}$$

where $p \in P(A,B)$ and $q \in P$. From (5.2.12),

$$(5.2.13) \qquad g_N(z) = \frac{zf'(z)}{p(z)}$$

$$(5.2.14) \qquad \frac{f'(z)}{g'(z)} = \frac{p(z)}{q(z)}\ .$$

As in the proof of Theorem 5.2.1, replace z by $\epsilon^j z$ in (5.2.12) to obtain

$$(5.2.15) \qquad g_N'(z) = \frac{g'(z)}{q(z)}\ [\frac{1}{N}\sum_{j=0}^{N-1} q(\epsilon^j z)].$$

Differentiation of (5.2.13) on both sides yields

$$(5.2.16) \quad g_N'(z) = -p'(z)[zf'(z)/p(z)^2] + [zf''(z) + (f'(z)/p(z))].$$

Using (5.2.14), (5.2.15) becomes

(5.2.17) $\qquad g_N'(z) = \dfrac{f'(z)}{p(z)} \, [\dfrac{1}{N} \sum\limits_{j=0}^{N-1} q(\varepsilon^j z)].$

Comparing (5.2.16) and (5.2.17), we obtain

$$\dfrac{f''(z)}{f'(z)} = \dfrac{p'(z)}{p(z)} + \dfrac{1}{Nz} \, (\sum\limits_{j=0}^{N-1} q(\varepsilon^j z) - N).$$

After a repeated integration we easily get (5.2.10) and by Theorem 5.2.1 we obtain (5.2.11).

Sufficiency part of this theorem can be proved on the same lines as those of Theorem 5.2.1. So we omit its proof.

REMARK 5.2.2 : For $A = 1-2\beta$ $(0 \le \beta < 1)$ and $B = -1$, the above theorem gives a structural formula for a function to be N-close-to-convex of order β in U and for the case $N = 1$ with $\beta = 0$ reduces to the well known structural formula for a close-to-convex (univalent) function in U.

5.3 CONVOLUTION THEOREMS :

Using the convolution techniques we give necessary and sufficient conditions for a function f in H to be in $S_N^*(A,B)$ and $K_N(A,B)$.

THEOREM 5.3.1 : A function $f \in H$ is in $S_N^*(A,B)$ if and only if,

(5.3.1) $\dfrac{1}{z}[f(z) * \dfrac{z + \{1+Ax+(1+Bx) \, A_{N-1}(z)\}(B-A)^{-1}x^{-1}z^2}{(1-z)^2 \, A_N(z)}] \ne 0,$

$$z \in U , \; |x| = 1$$

where $\quad A_N(z) = (1-z^N)/(1-z)$.

Proof : A function $f \in H$ is in $S_N^*(A,B)$ if and only if

$$\frac{zf'(z)}{f_N(z)} \neq \frac{1+Ax}{1+Bx} \text{ for } z \in U \text{ and } |x| = 1$$

which because of normalization of f is equivalent to

(5.3.2) $\quad \frac{1}{z} [(1+Bx)zf'(z) - f_N(z)(1+Ax)] \neq 0, \ z \in U.$

Since $zf'(z) = f(z) * \dfrac{z}{(1-z)^2}$ and $f_N(z) = f(z) * \dfrac{z}{(1-z^N)}$,

(5.3.2) on simplification reduces to

$$\frac{1}{z} \left[f(z) * \frac{z+\{1+Ax+(1+Bx)(1-z^{N-1})(1-z)^{-1}\}(B-A)^{-1}x^{-1}z^2}{(1-z)(1-z^N)} \right] \neq 0$$

for $z \in U$, $|x| = 1$, which is the desired convolution condition.

REMARK 5.3.1 : For $N = 1$, Theorem 5.3.1 yields the results found in [130].

Putting $N = 2$, $A = 1$ and $B = -1$ in Theorem 5.3.1, we obtain the following convolution condition for the univalent functions starlike with respect to symmetric points considered by Sakaguchi [120].

COROLLARY 5.3.1 : A function f in H is univalent starlike with respect to symmetric points in U if and only if,

$$\frac{1}{z} \left[f(z) * \frac{z - \dfrac{z^2}{x}}{(1-z)^2(1+z)} \right] \neq 0, \ z \in U, \ |x| = 1.$$

THEOREM 5.3.2 : A function f in H is in $K_N(A,B)$ if and only if for all z in U and all x, $|x| = 1$,

$$(5.3.2) \quad \frac{1}{z}\left[f(z) * \frac{z+\{(1+Ax)D_N(z)+(1+Bx)E_N(z)\}(B-A)^{-1}x^{-1}z^2}{(1-z)^3 A_N^2(z)}\right] \neq 0$$

where

$$(5.3.3) \quad A_N(z) = (1-z^N)/(1-z)$$

$$(5.3.4) \quad D_N(z) = (N-1)(1-z)\{(1-z)A_{N-1}(z)-1\} + 1$$

$$(5.3.5) \quad E_N(z) = A_{N-1}(z) \cdot (1+z)\{A_N(z)+1\} + 1.$$

Proof : Set $g(z) = \dfrac{z+\{(1+Ax)+(1+Bx)A_{N-1}(z)\}(B-A)^{-1}x^{-1}z^2}{(1-z)^2 A_N(z)}$

where $A_N(z)$ is defined by (5.3.3). Using the identity $zf'(z) * g(z) = f(z) * zg'(z)$ and the fact that $f \in K_N(A,B)$ if and only if $zf' \in S_N^*(A,B)$, the result follows from Theeorem 5.3.1.

REMARK 5.3.2 : Putting N = 1 in the above theorem, we obtain the necessary and sufficient condition, in terms of convolution, for a function to be in K(A,B) obtained recently by Silverman and Silvia [130]. For N = 1, $A = 1-2\beta$ ($0 \leq \beta < 1$) and B = -1 in Theorem 5.3.2 we obtain the results found in [131].

5.4 GENERALIZATION OF POLYA-SCHOENBERG CONJECTURE :

We next show that Polya-Schoenberg conjecture is valid for the classes introduced in this chapter. More

precisely we have

THEOREM 5.4.1 : If $\varphi \in K$ <u>and</u> $f \in S_N^*(A,B)$ $(K_N(A,B)$ <u>or</u> $C_N(A,B)$ <u>resp.</u>) <u>then</u> $\varphi * f \in S_N^*(A,B)$ $(K_N(A,B)$ <u>or</u> $C_N(A,B)$ <u>resp.</u>).

<u>Proof</u> : We shall prove that if $\varphi \in K$ and $f \in S_N^*(A,B)$ then $\varphi * f \in S_N^*(A,B)$, other cases can be treated in a similar manner. Suppose $f \in S_N^*(A,B)$, then by definition we have

$$(5.4.1) \qquad \frac{zf'}{f_N} \prec \frac{1+Az}{1+Bz} \quad (z \in U, \ -1 \leqq B < A \leqq 1).$$

Since $|z| = 1$ is mapped by $(1+Az)/(1+Bz)$ onto a circle centered at $(1-AB)/(1-B^2)$ with radius $(A-B)/(1-B^2)$, lying in the right half plane, it follows that from (5.4.1) that $f \in S_N^*$. So changing z to $\varepsilon^j z$ in (5.4.1) and adding the resultant N equations obtained for $j = 0,1,2,\ldots,N-1$, we have that $f_N \in S^*(0) \equiv S^*$.

Since $z(\varphi * f)'(z) = \varphi(z) * zf'(z)$ and

$$(\varphi * f)_N(z) = \varphi(z) * f_N(z), \text{ we have}$$

$$\frac{z(\varphi * f)'(z)}{(\varphi * f)_N(z)} = \frac{\varphi(z) * \dfrac{zf'(z)}{f_N(z)} \cdot f_N(z)}{\varphi(z) * f_N(z)}.$$

By Lemma 2.3.1, the range of $z(\varphi * f)'(z)/(\varphi * f)_N(z)$ lies in the closed convex hull of $(zf'(z)/f_N(z))(U)$. The theorem, now follows from (5.4.1).

<u>REMARK 5.4.1</u> : For $N = 1$ above theorem reduces to the celebrated theorem of Ruscheweyh and Sheil-Small [117] on

Polya-Schoenberg conjecture.

(ii)   For $A = 1-2\beta$ ($0 \leq \beta < 1$) and $B = -1$ in Theorem 5.4.1 we obtain the results for the classes $S_N^*(\beta)$ ($K_N(\beta)$ or $C_N(\beta)$ resp.).

(iii)   If we take $A = 1$, $B = -1$ and $N = 2$ we obtain the corresponding results for univalent function starlike with respect to symmetric points.

(iv)   For the case $N = 1$, Theorem 5.4.1 would lead to the results for the classes $S^*(A,B)$, $K(A,B)$ and $C(A,B)$ respectively.

(v)   Taking φ to be the convex function $\sum_{n=1}^{\infty} \frac{c+1}{c+n} z^n$ (Re $c > -1$), we obtain that, if $f \in S_N^*(A,B)$($K_N(A,B)$ or $C_N(A,B)$ resp.) the integral transform F_c defined by (1.6.3) namely

$$F_c(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t)dt = f(z) * \{\sum_{n=1}^{\infty} \frac{c+1}{c+n} z^n\}$$

is also in $S_N^*(A,B)$ ($K_N(A,B)$ or $C_N(A,B)$ resp.).   This includes the result of Bernardi [8], Bajpai and Srivastava [6], Lewandowski et al. [58] and the recent result of Silverman and Silvia [130, Corollary 1] obtained for $N = 1$.

(vi)   If $f \in S_N^*(A,B)$($K_N(A,B)$ or $C_N(A,B)$ resp.), taking φ to be the convex function $\frac{1}{1-x} \log (\frac{1-xz}{1-z})$, $|x| \leq 1$, $x \neq 1$, then the integral transform of f defined by

$$\int_0^z \frac{f(t)-f(xt)}{t-xt} dt = f(z) * \{\frac{1}{1-x} \log (\frac{1-xz}{1-z})\}$$

is also seen to be in $S_N^*(A,B)$ $(K_N(A,B)$ or $C_N(A,B)$ resp.). This includes the recent results of Silverman and Silvia [130, Corollary 2] obtained for $N = 1$.

__DEFINITION 5.4.1__ : __The__ __functions__ f_1 __and__ f_2 __in H,__ __are__ __called__ __mutually__ __adjoint__ __if__ __they__ __satisfy__

$$(5.4.2) \qquad \text{Re } \{ \frac{zf_i'(z)}{f_1(z)+f_2(z)} \} > 0, \; z \in U \text{ for } i = 1,2.$$

This definition is due to Lewandowski and Stankiewicz [59]. It may be noted from (5.4.2) that f_1 and f_2 are close-to-convex (univalent) in U with respect to the starlike function $(f_1+f_2)/2$.

Now the remarks 5.4.1(iv) and (v) give

__COROLLARY 5.4.1__ : __If the__ __functions__ f_1 __and__ f_2 __in H__ __are__ __mutually__ __adjoint__ __so__ __also__ F_1 __and__ F_2 __(or__ H_1 __and__ H_2 __resp.)__ __where__

$$(5.4.3) \qquad F_i(z) = \frac{1+c}{z^c} \int_0^z f_i(t) \, t^{c-1} \, dt, \; \text{Re } c > -1$$

__and__

$$H_i(z) = \int_0^z \frac{f_i(t)-f_i(xt)}{t-xt} \, dt, \quad |x| \leq 1, \; x \neq 1$$

__for__ $i = 1,2$.

__Proof__ : Suppose that f_1 and f_2 are mutually adjoint then

$$\text{Re } \{ \frac{zf_i'(z)}{g(z)} \} > 0, \; z \in U, \; i = 1,2$$

where g defined by $g(z) = [f_1(z)+f_2(z)]/2$ is close-to-convex

in U. This shows that $f_i \in C_1(1,-1) \equiv C$ for $i = 1,2$. Thus by Remark 5.4.1(iv) F_i defined by (5.4.3) for $i = 1,2$ is also in C. Further $g = (f_1+f_2)/2 \in S^*$ implies $(F_1+F_2)/2$ is also in S^*. This shows that F_1 and F_2 are also mutually adjoint follows from the Remark 5.4.1(iv) and the fact that f_1 and f_2 are mutually adjoint. Similar arguments can be used for H_i, $(i = 1,2)$. This completes the proof of Corollary. One can generalize the above result as follows.

We define the functions $f_1, f_2, \ldots, f_n \in H$ such that $\{f_1, \ldots, f_n\}$ is said to be in $\Phi_n(A,B)$ $(-1 \leq B < A \leq 1)$ if for each $j = 1, 2, \ldots, n$,

$$(5.4.4) \quad p_j(z) = \frac{\displaystyle\sum_{i=1}^{n} \alpha_i (z\, f_j'(z))}{\displaystyle\sum_{i=1}^{n} \alpha_i\, f_i(z)} \in P(A,B) \quad (z \in U, -1 \leq B < A \leq 1)$$

where α_i's are any fixed positive real numbers.

It may be noted that (5.4.4) gives by the definition of $P(A,B)$ that

$$(5.4.5) \quad \frac{1}{\displaystyle\sum_{i=1}^{n} \alpha_i} \left[\sum_{j=1}^{n} \alpha_j\, p_j(z) \right] \in P(A,B)$$

and further it can be easily seen that $\dfrac{1}{\displaystyle\sum_{i=1}^{n} \alpha_i}\left[\displaystyle\sum_{i=1}^{n} \alpha_i f_i \right]$ is in $S^*(A,B)$ and so each f_j for $j = 1, 2, \ldots, n$, is close-to-convex (univalent) in U.

For A = 1, B = -1, j = 1,2 (i.e., with n = 2), $\alpha_1 = \alpha_2 = 1$ the class $\Phi_2(1,-1)$ reduces to the definition of mutually adjoint of f_1 and f_2. Now the remarks 5.4.1(v) and (vi) give the following corollary and can be proved in a manner similar to that of the above corollary.

COROLLARY 5.4.2 : Let $f_i \in H$ (i = 1,2,...,n) and $\{f_1,\ldots,f_n\} \in \Phi_n(A,B)$.

Then

(i) $\{F_1, F_2, \ldots, F_n\} \in \Phi_n(A,B)$

(ii) $\{H_1, H_2, \ldots, H_n\} \in \Phi_n(A,B)$

where

$$F_i(z) = \frac{1+c}{z^c} \int_0^z f_i(t) \, t^{c-1} \, dt, \quad \text{Re } c > -1$$

and

$$H_i(z) = \int_0^z \frac{f_i(t) - f_i(xt)}{t - xt} \, dt, \quad |x| \leq 1, \; x \neq 1$$

for i = 1,2,...,n.

The de la Vallée poussin mean of function f in H defined by $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$ is the polynomial defined by

$$V_n(z,f) = \frac{(n!)^2}{(2n)!} \sum_{j=1}^{n} \frac{(2n)!}{(n-j)! \, (n+j)!} a_j z^j$$

$$= \frac{n}{n+1} z + \frac{n(n-1)}{(n+1)(n+2)} a_2 z^2 + \ldots + \frac{n(n-1)\ldots 2.1}{(n+1)(n+2)\ldots(2n)} a_n z^n$$

It was shown in [98] that for every positive integer n, $V_n(z,f)$ is convex or starlike in U whenever f is convex or

starlike in U. Since k defined by $k(z) = z/(1-z)$ is convex in U it follows that $V_n(z,k)$ is also convex in U. Using this fact we prove

THEOREM 5.4.2 : _If_ $f \in S_N^*(A,B)$ $(K_N(A,B)$ **or** $C_N(A,B)$ resp.) _then the de la Valleé poussin mean_ $V_n(z,f)$ _is also in_ $S_N^*(A,B)$ $(K_N(A,B)$ _or_ $C_N(A,B)$ resp.)

Proof : We prove that $f \in S_N^*(A,B)$ implies $V_n(z,f) \in S_N^*(A,B)$; the other cases can be dealt with in a similar fashion. Applying Lemma 2.3.1 with $\varphi(z) = V_n(z,k)$ and $k(z) = z/(1-z)$, we see that

$$V_n(z,k) * zf'(z) = z(V_n(z,f))'$$

$$V_n(z,k) * f_N(z) = (V_n(z,f))_N \text{ , and so}$$

$$\frac{z(V_n(z,f))'}{(V_n(z,f))_N} = \frac{V_n(z,k) * \dfrac{zf'(z)}{f_N(z)} f_N(z)}{V_n(z,k) * f_N(z)}$$

takes values in the convex hull of $(zf'/f_N)(U)$. Thus it follows that $V_n(z,f)$ is in $S_N^*(A,B)$.

In order to prove next two theorems we use Lemma 2.3.1

THEOREM 5.4.3 : _If_ $f \in S_N^*(1/2)$, _then the de la Vallée poussin mean_ $V_n(z,f)$ _satisfies_

(5.4.4) $\mathrm{Re}\ \{\dfrac{f_N(z)}{V_1(z,f)}\} > 1, \ z \in U$

(5.4.5)   Re $\{\dfrac{f_N(z)}{(V_n(z,f))_N}\} > 0$, $z \in U$, $n \geq 2$.

Proof : Since $f \in S_N^*(1/2)$, f_N is in $S^*(1/2)$ and so

Re $\{\dfrac{f_N(z)}{z}\} > \dfrac{1}{2}$, $z \in U$ [143], which is equivalent to (5.4.4).

Since both $f_N(z) = z + \sum\limits_{m=1}^{\infty} a_{Nm+1} z^{Nm+1}$ and

$(V_n(z,f))_N = V_n(z,f) * f_N(z)$ have the power series expansion

of the same form, it is enough to prove the result for n of the

form $n = Nm+1$, where $m \geq 0$ is any integer.

To prove (5.4.5) we note that

$V_{Nm+1}(z,f) * f_N(z)$

$= \dfrac{Nm+1}{Nm+2}z + \dots + \dfrac{(Nm+1)(Nm) \dots 2.1}{(Nm+2) \dots (2(Nm+1))} a_{Nm+1} z^{Nm+1}$

$= (V_{Nm+1}(z,f))_N$

and

$F_n(z) = \dfrac{3n}{(n+1)(n+2)} (1-z) + \dfrac{5n(n-1)}{(n+1)(n+2)(n+3)} (1-z^2) + \dots$

$\qquad + \dfrac{(2n+1)n(n-1) \dots 2.1}{(n+1)(n+2) \dots (2n+1)} (1-z^n)$

satisfies Re $\{F_n(z)\} > 0$ in U with

$\qquad V_{Nm+1}(z,f) = f(z) * \dfrac{z}{1-z} F_{Nm+1}(z).$

Also by Lemma 2.3.1

$$\frac{f_N(z) \; * \; \frac{z}{1-z} \; F_{Nm+1}(z)}{f_N(z) \; * \; \frac{z}{1-z}} = \frac{f_N(z) \; * \; V_{Nm+1}(z,f)}{f_N(z)}$$

$$= \frac{(V_{Nm+1}(z,f))_N}{f_N(z)}$$

takes values in the convex hull of $F_{Nm+1}(U)$.

This completes the proof of (5.4.5).

The case $N = 1$ is due to Ruscheweyh and Sheil-Small [117] whereas $N = 2$ gives

COROLLARY 5.4.1 :   If f satisfies

$$\text{Re } \{ \frac{zf'(z)}{f(z)-f(-z)} \} > \frac{1}{4} \; , \; z \in U$$

then

(i)   Re $\{ \frac{f(z)-f(-z)}{z} \} > 1$

(ii)   Re $\{ \frac{f(z)-f(-z)}{V_n(z,f)-V_n(-z,f)} \} > 0$

for each integer $n > 1$.

THEOREM 5.4.4 :   If $f \in S_N^*(1/2)$, then for each integer n we have

$$\text{Re } \{ \frac{f_N(z)}{(s_n(z,f))_N} \} > \frac{1}{2} \; , \; z \in U$$

where $s_n(z,f)$ denotes the nth partial sum of f.

Proof :   As mentioned in Theorem 5.4.3, clearly it is enough to prove the theorem for $n = Nm+1$, where m is any positive integer.   Since $f \in S_N^*(1/2)$ (and so $f_N \in S^*(1/2)$) taking $\varphi(z) = f_N(z)$, $g(z) = z/(1-z)$ and $F(z) = 1-z^{Nm+1}$ in Lemma 2.3.1

we have that

$$\frac{f_N(z) \ast \frac{z}{1-z}(1-z^{Nm+1})}{f_N(z) \ast \frac{z}{1-z}} = \frac{(s_{Nm+1}(z,f))_N}{f_N(z)}$$

$$= \frac{s_{Nm+1}(z,f_N)}{f_N(z)}$$

takes values in the convex hull of $F(U)$ i.e. in $|w-1| < 1$.
Therefore

$$\left| \frac{s_{Nm+1}(z,f_N)}{f_N(z)} - 1 \right| < 1, \ z \in U.$$

Hence the theorem.

The case $N = 1$ was obtained in [117] and the case
$N = 2$ gives

COROLLARY 5.4.2 : If f satisfies

$$\text{Re} \ \{ \frac{zf'(z)}{f(z)-f(-z)} \} > \frac{1}{4} \ , \ z \in U$$

then the relation

$$\text{Re} \ \{ \frac{f(z)-f(-z)}{s_n(z,f) - s_n(-z,f)} \} > \frac{1}{2}$$

holds for each integer $n \geq 1$.

In order to obtain some results concerning the nth
partial sums of functions in $S_N^*(A,B)$ etc., we need the
following lemma due to S. Singh [139] (see also [41])

LEMMA 5.4.3 : For the convex function $k(z) = z/(1-z)$, $z \in U$
and for each $n \geq 2$, the nth partial sum $s_n(z,k)=z(1-z^n)/(1-z)$
of k is convex at least in the disc $|z| < r_n$ where r_n

($\geq \frac{1}{4}$, for every n) is the smallest positive root of the equation

$$(5.4.6) \quad l'(r,n) \equiv 1-r-(n+1)^2 r^n - (2n^2+2n-1)r^{n+1} - n^2 r^{n+2} = 0.$$

THEOREM 5.4.5 : If f be in $S_N^*(A,B)$ ($K_N(A,B)$ or $C_N(A,B)$ resp.), then the nth partial sum $s_n(z,f)$ is also in $S_N^*(A,B)$($K_N(A,B)$ or $C_N(A,B)$ resp.) atleast for z satisfying $|z| < r_n$ where r_n is the smallest positive root of the equation (5.4.6).

Proof : We shall prove the theorem for the case $f \in S_N^*(A,B)$ the other cases being treated in a similar manner. Let $f \in S_N^*(A,B)$ and $k(z) = z/(1-z)$. Since k is convex univalent in U and $s_n(z,k) = z(1-z^n)/(1-z)$, applying the above lemma we have that $s_n(z,k)$ is also convex univalent atleast in $|z| < r_n$, where r_n is the smallest positive root of the equation (5.4.6). Since $f \in S_N^*(A,B)$ implies $f_N \in S^*$, applying Lemma 2.3.1 with $\varphi(z) = s_n(z,k)$, and $q(z) = f_N(z)$ we get that

$$\frac{s_n(z,k) * \frac{zf'(z)}{f_N(z)} \cdot f_N(z)}{s_n(z,k) * f_N(z)} = \frac{z(s_n(z,f))'}{(s_n(z,f))_N} \quad , \quad |z| < r_n$$

is in the convex hull of (zf'/f_N) for $|z| < r_n$.

or $\quad \dfrac{z(s_n(z,f))'}{(s_n(z,f))_N} \prec \dfrac{1+Az}{1+Bz} , \quad |z| < r_n.$

Hence the theorem.

5.5 NEIGHBORHOODS OF FUNCTIONS RELATED TO $S_N^*(A,B)$:

For a function $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$, Ruscheweyh [116] introduced the concept of neighborhood $Q_\delta(f)$ of functions in H as follows :

$$(5.5.1) \quad Q_\delta(f) = \{g \in H : g(z) = z + \sum_{k=2}^{\infty} b_k z^k \text{ and } \sum_{k=2}^{\infty} k |a_k - b_k| \leq \delta\}$$

where $0 < \delta < \infty$.

He proved, among other results, that if $f \in H$, $\delta > 0$, and for all $\eta \in ¢$, with $|\eta| < \delta$,

$$\frac{f(z) + \eta z}{1 + \eta} \in S^*$$

then $Q_\delta(f) \subset S^*$. Fournier [33], Rahman and Stankiewicz [102], Brown [14] and Rajasekaran [103] also obtained some interesting results on neighborhoods of univalent functions.

To find some results for the class $S_N^*(A,B)$ analogous to those obtained by Ruscheweyh [116] we introduce the concept of neighborhood as follows :

DEFINITION 5.5.1 : For $-1 \leq B < A \leq 1$ and $\delta \geq 0$ we define $Q_{A,B,\delta}(f,N)$ the neighborhood of a function $f \in H$,
$f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ as

(5.5.2) $\quad Q_{A,B,\delta}(f,N) = \{g \in H : g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$ and

$$d(f,g) = \sum\limits_{k=2}^{\infty} \left(\frac{k - \delta_k + |Bk - A\delta_k|}{A-B} \right) |b_k - a_k| \le \delta\}$$

where

$$\delta_n = \begin{cases} 1 & \underline{\text{if}}\ n = Nj+1,\ j = 1,2,\ldots \\ 0 & \underline{\text{otherwise}} \end{cases}$$

and N inside the bracket is a fixed positive integer.

Note that $Q_{1,-1,\delta}(f,1) = Q_{1,0,\delta}(f,1) = Q_\delta(f)$ and so in this case (5.5.2) reduces to (5.5.1) defined by Ruscheweyh [116].

Now we have

THEOREM 5.5.1 : Let $f \in H$ and for all complex numbers η such that $|\eta| < \delta$, suppose

(5.5.3) $\qquad \dfrac{f(z) + \eta z}{1+\eta} \in S_N^*(A,B);$

Then $\qquad Q_{A,B,\delta}(f,N) \subset S_N^*(A,B).$

Proof : Let

$$H_N^*(A,B) = \{h_\Theta \in H : h_\Theta(z) = \frac{(1+Be^{i\Theta})\dfrac{z}{(1-z)^2} - (1+Ae^{i\Theta})\dfrac{z}{1-z^N}}{(B-A)e^{i\Theta}}, \Theta \in (0,2\pi)\}$$

By Theorem 5.3.1, we have

(5.5.4) $\quad f \in S_N^*(A,B)$ if and only if $\frac{1}{z}[(f * h_\Theta)(z)] \ne 0$

for all $h_\Theta \in H_N^*(A,B)$ and $z \in U.$

Also, if $h_\Theta(z) = z + \sum\limits_{k=2}^{\infty} h_k z^k \in H_N^*(A,B)$, then for $n = 2,3,\ldots$

$$h_n = \frac{n - \delta_n + (Bn - A \delta_n)e^{i\Theta}}{(B-A)e^{i\Theta}}$$

where

$$\delta_n = \begin{cases} 1 & \text{if } n = Nj+1, \; j = 1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $|h_n| \leqq \dfrac{n - \delta_n + |Bn-A \delta_n|}{A-B}$, $-1 \leqq B < A \leqq 1$.

Let f defined by $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ in H satisfy (5.5.3) for all $|\eta| < \delta$. Then we obtain from (5.5.4) for $h_\Theta \in H_N^*(A,B)$, that

$$\frac{1}{z} \left[\frac{(f * h_\Theta)(z) + \eta z}{1+\eta} \right] \neq 0, \; z \in U, \; |\eta| < \delta$$

or equivalently

$$(5.5.5) \qquad \left| \frac{(f * h_\Theta)(z)}{z} \right| \geqq \delta , \; z \in U.$$

Observe that (5.5.5) is equivalent to (5.5.3).
Now we assume that g defined by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, is in $Q_{A,B,\delta}(f,N)$. Then for $h_\Theta \in H_N^*(A,B)$,

$$\left| \frac{(g * h_\Theta)(z)}{z} \right| = \left| \frac{(f * h_\Theta)(z)}{z} + \frac{((g-f) * h_\Theta)(z)}{z} \right|$$

$$\geqq \delta - \left| \frac{((g-f) * h_\Theta)(z)}{z} \right|.$$

But for z ∈ U,

$$\left|\frac{((g-f) * h_\Theta)(z)}{z}\right| = \left|\sum_{k=2}^{\infty} (b_k - a_k) h_k z^k\right|$$

$$\leq |z| \sum_{k=2}^{\infty} \left(\frac{k - \delta_k + |Bk - A\delta_k|}{A - B}\right) |b_k - a_k|$$

$$< \delta.$$

We therefore obtain $\dfrac{(g * h_\Theta)(z)}{z} \neq 0$ in U for all

$h_\Theta \in H_N^*(A,B)$ and so $g \in S_N^*(A,B)$. Hence, the theorem follows.

REMARK 5.5.1 : Putting N = 1, A = 1 and B = -1 in Theorem 5.5.1 we obtain a result of Ruscheweyh [116] whereas for A = 1-2β (0 ≤ β < 1) and B = -1 a result of Reddy [104] is obtained. The case N = 1 of Theorem 5.5.1 leads to a result of Rajasekaran [103].

THIRD ORDER DIFFERENTIAL INEQUALITIES IN THE COMPLEX PLANE

6.1   INTRODUCTION :

Let w be analytic in U, with $w(0) = 0$, and let $h(r,s)$ be a complex valued function defined in a domain contained in $\mathbb{C}^2$. With some simple conditions on h it was shown by Miller [73] that

$$(6.1.1) \quad |h(w(z),zw'(z))| < 1 \text{ implies } |w(z)| < 1, \text{ for } z \in U.$$

In particular

$$(6.1.2) \quad |w(z)+zw'(z)| < 1 \text{ implies } |w(z)| < 1, \text{ for } z \in U.$$

The above result was further extended by Miller and Mocanu [74] to functions $h(r,s,t)$ defined in a domain contained in $\mathbb{C}^3$ where conditions on h were obtained to ensure that

$$(6.1.3) \quad |h(w(z),zw'(z),z^2w''(z))| < 1 \text{ implies } |w(z)| < 1, \text{ for } z \in U$$

and

$$(6.1.4) \quad \text{Re } \{h(w(z),zw'(z),z^2w''(z))\} > 0 \text{ implies Re } w(z) > 0,$$
$$\text{for } z \in U.$$

In particular, it was shown that

$$(6.1.5) \quad |w(z)+zw'(z)+z^2w''(z)| < 1 \text{ implies } |w(z)| < 1, \text{ for } z \in U.$$

Motivated by (6.1.2) and (6.1.5), Miller in [21 , Prob. 5.61]
asked if, for n = 3,4,...,

(6.1.6) $|w(z)+zw'(z) + ...+ z^n w^{(n)}(z)| < 1$ for z $\in$ U

would imply $|w(z)| < 1$ for z $\in$ U. In [147], Toppila gave
an affirmative answer to this problem by showing that if
w be analytic in U with w(0) = 0, satisfying (6.1.6) for
some n $\geq$ 3, then $|w(z)| < 71/80$ in U. Very recently
Yueqing [152] improved the result of Toppila by showing
that (6.1.6) infact implies

(6.1.7) $|w(z)| < \sqrt{\sum_{k=1}^{n} (1/p_k^2)} < 0.554$, for z $\in$ U

where

$$p_k = \begin{cases} 1+k+k(k-1)+...+k! \text{ when } k \leq n \\ 1+k+k(k-1)+...+k(k-1)...(k-n+1) \text{ when } k > n. \end{cases}$$

Goldstein et al. [38] while solving a general problem than
the one proposed by Miller stated that, the best bound
in (6.1.7) is likely to be 1/2. However this is yet to be
established.

In this chapter an attempt has been made to generali-
the results of Miller and Mocanu [73,74] viz., (6.1.2) and
(6.1.3). In Section 6.2 two lemmas have been proved to
generalize the Jack-Miller-Mocanu Lemma. These lemmas have

been used to generate subclasses of bounded functions that would demonstrate that certain third order differential equations have bounded solutions. In Section 6.3, further use of these lemmas leads to results concerning the third order differential equations having solutions with positive real part.

6.2   GENERALISATION OF JACK-MILLER-MOCANU LEMMA :

The basic tools in proving our results are the following lemmas.

LEMMA 6.2.1 :   Let g defined by $g(z) = g_n z^n + g_{n+1} z^{n+1} + \ldots$ be analytic in U, with $g_n \neq 0$, and let $z_0 \neq 0$, $z_0 = r_0 e^{i\Theta_0}$ $(0 < r_0 < 1)$ be a point of U such that

$$(6.2.1) \qquad |g(z_0)| = \max_{|z| \leq |z_0|} |g(z)|.$$

Then there is a real number m, $m \geq n \geq 1$, such that

$$(6.2.2) \qquad \frac{z_0 g'(z_0)}{g(z_0)} = m$$

$$(6.2.3) \qquad \text{Re } \{1 + \frac{z_0 g''(z_0)}{g'(z_0)}\} \geq m.$$

Further if

$$(6.2.4) \qquad \frac{\partial^3 \text{ arg } g(z)}{\partial \Theta^3} \Big|_{z=z_0} \leq 0$$

<u>then</u>

$$(6.2.5) \qquad \text{Re} \left\{ \frac{z_0 g'(z_0) + 3z_0^2 g''(z_0) + z_0^3 g'''(z_0)}{z_0 g'(z_0)} \right\} \geqq m^2.$$

<u>Proof</u> : Although (6.2.2) and (6.2.3) are known and together form the Jack-Miller-Mocanu Lemma [74], we reproduce the proof for the sake of completeness. For the proof of (6.2.5) we follow the method similar to that of Miller and Mocanu [74]. If we let $g(z) = R(r_0, \Theta) e^{i\Phi(r_0, \Theta)}$ for $z = r_0 e^{i\Theta}$, then

$$(6.2.6) \qquad \frac{z g'(z)}{g(z)} = \frac{\partial \Phi}{\partial \Theta} - \frac{i}{R} \frac{\partial R}{\partial \Theta} .$$

Since $|g(z_0)|$ is a maximum value, we conclude that

$$\text{Re} \left\{ \frac{\partial}{\partial \Theta} \ln g(z) \right\} = 0 \text{ and } \text{Re} \left\{ \frac{\partial}{\partial r} \ln g(z) \right\} \geqq 0$$

hold for $z = z_0$.

The first relation implies that $[z_0 g'(z_0)/g(z_0)]$ is real, while the second relation implies that $[z_0 g'(z_0)/g(z_0)]$ is non-negative. Now $g(z)$ can be written in the form $g(z) = z^n b(z)$, so that $|b(z_0)| = \max_{|z| \leqq |z_0|} |b(z)|$. Hence $[z_0 b'(z_0)/b(z_0)]$ is real and non-negative; however,

$$\frac{z_0 g'(z_0)}{g(z_0)} = n \left[1 + \frac{z_0 b'(z_0)}{b(z_0)} \right] = m \geqq n \geqq 1.$$

which establishes (6.2.2).

Differentiating (6.2.6) with respect to Θ we obtain

$$(6.2.7) \quad i\left[\frac{zg'(z)}{g(z)}\left\{\frac{z(zg'(z))'}{zg'(z)} - \frac{zg'(z)}{g(z)}\right\}\right]$$

$$= \frac{\partial^2 \Phi}{\partial\Theta^2} - i\left[\frac{1}{R}\frac{\partial^2 R}{\partial\Theta^2} - \frac{1}{R^2}\left(\frac{\partial R}{\partial\Theta}\right)^2\right].$$

Since $|g(z_0)|$ is a maximum value, we have $\frac{\partial R(z_0)}{\partial\Theta} = 0$,

$\frac{\partial^2 R(z_0)}{\partial\Theta^2} \leqq 0$ and so .

$$0 \geqq \frac{\partial^2}{\partial\Theta^2}\{Re(\ln g(z_0)\}$$

$$= Re\left[\frac{z_0 g'(z_0)}{g(z_0)}\left\{(1 + \frac{z_0 g''(z_0)}{g'(z_0)}) - \frac{z_0 g'(z_0)}{g(z_0)}\right\}\right]i^2,$$

which with (6.2.2) yields (6.2.3). This proves (6.2.2) and (6.2.3).

Now we proceed to prove (6.2.4). Differentiation of (6.2.7) with respect to Θ , leads to

$$\frac{z(z(zg'(z))')'}{zg'(z)} \cdot \frac{zg'(z)}{g(z)} - 3z\frac{(zg'(z))'}{zg'(z)}\left(\frac{zg'(z)}{g(z)}\right)^2 + 2\left(\frac{zg'(z)}{g(z)}\right)^3$$

$$(6.2.8) \qquad = \frac{-\partial^3\Phi}{\partial\Theta^3} + i\frac{\partial}{\partial\Theta}\left[\frac{1}{R}\frac{\partial^2 R}{\partial\Theta^2} - \frac{1}{R^2}\left(\frac{\partial R}{\partial\Theta}\right)^2\right].$$

By (6.2.4), we see that the real part of left hand side of (6.2.8) is non-negative and so using (6.2.2) we get

$$Re\left\{\frac{z_0^3 g'''(z_0) + 3z_0^2 g''(z_0) + z_0 g'(z_0)}{z_0 g'(z_0)}\right\}m - 3\,Re\left\{1 + \frac{z_0 g''(z_0)}{g'(z)}\right\}m^2 + 2m^3$$

Now using (6.2.3) we obtain (6.2.5). Hence the proof of

the Lemma 6.2.1.

REMARK 6.2.1 : The set of all functions satisfying the conditions of Lemma 6.2.1 is not vacuous. For instance, consider $g(z) = \dfrac{z}{1-z^2}$. Then

$$R = |g(re^{i\theta})| = |g(r,\theta)| = r/\{1-2r^2 \cos 2\theta + r^4\}^{1/2}$$

$$\frac{\partial R}{\partial \theta} = -2r^3 \sin 2\theta/(1-2r^2 \cos 2\theta + r^4)^{3/2}.$$

Therefore

$$\left.\frac{\partial R}{\partial \theta}\right|_{\theta=0} = 0$$

and

$$\left.\frac{\partial^2 R}{\partial \theta^2}\right|_{\theta=0} = -4r^3/(1-r^2)^3 < 0 .$$

Hence, $|g(r_0)| = \max\limits_{|z| \leqq |z_0|=r_0} |g(r,\theta)|.$

Again, if we set

$$\Phi = \arg (z/(1-z^2))$$

i.e., $\quad \Phi(\theta) = \theta - \tan^{-1} \left(\dfrac{-r_0^2 \sin 2\theta}{1-r_0^2 \cos 2\theta}\right)$

then $\quad \dfrac{\partial \Phi}{\partial \theta} = \dfrac{1-r_0^4}{1-2r_0^2 \cos 2\theta + r_0^4}$,

$$\frac{\partial^2 \Phi}{\partial \theta^2} = -(1-r_0^4) \frac{4r_0^2 \sin 2\theta}{(1-2r_0^2 \cos 2\theta + r_0^4)^2} , \text{ and}$$

$$\frac{\partial^3 \Phi}{\partial\theta^3} = -(1-r_o^4)\ \{\frac{(4r_o^2)\ 2\ \cos\ 2\theta}{(1-2r_o^2\ \cos\ 2\theta+r_o^4)^2}\} + \frac{(4r_o^2\ \sin\ 2\theta)^2(1-r_o^4)}{(1-2r_o^2\ \cos\ 2\theta+r_o^4)^3}\ .$$

This gives

$$\frac{\partial^3\ \arg\ g(r_o,\theta)}{\partial\theta^3}\Bigg|_{\theta=0} = \frac{-8r_o^2(1-r_o^4)}{(1-r_o^2)^4} < 0.$$

Hence the required conditions of Lemma 6.2.1 are satisfied. Similarly one may verify that the convex function $k(z) = z/(1-z)$ also satisfies the conditions of the above le

REMARK 6.2.2 : The following example shows that the condition (6.2.4) is not implied by the condition (6.2.1). So the condition (6.2.4) can not be dropped for the proof of (6.2.5). Consider

$$g(z) = z\ \frac{z+a}{1+az}\ \text{where}\ -1 < a < 1.$$

Here $\max\limits_{|z|\leq 1} |g(z)| = 1 = g(1)$. For $z = e^{i\theta}$ and θ near 0,

$$\Phi(\theta) = \arg\ g(z) = \text{Im}\ \log(e^{i\theta}+a)-\text{Im}\ \log\ (e^{-i\theta}+a),$$

$$\frac{\partial\Phi}{\partial\theta} = \text{Im}\ \{\frac{i\ e^{i\theta}}{e^{i\theta}+a}\} - \text{Im}\ \{\frac{-i\ e^{-i\theta}}{e^{-i\theta}+a}\}$$

$$= 2\ \text{Re}\ \{\frac{1}{1+a\ e^{i\theta}}\}$$

$$= \frac{1+a\ \cos\ \theta}{1+2a\ \cos\ \theta + a^2} = \alpha + \beta\theta^2 + \ldots$$

where

$$\alpha = \frac{1}{1+a} \text{ and } \beta = \frac{a}{2(1+a)^3}(1-a). \text{ Hence}$$

$$\frac{\partial^3 \Phi}{\partial \Theta^3} = 2\beta + \ldots \text{ and } \frac{\partial^3 \Phi}{\partial \Theta^3}\bigg|_{\Theta=0} = 2\beta.$$

Thus for $a < 0$, we have $\beta < 0$ whereas for $a > 0$, we have $\beta > 0$
This shows that nothing can be said about $\frac{\partial^3 \text{ arg } g(z)}{\partial \Theta^3}$ of the
point where $|g(z)|$ is maximal.

REMARK 6.2.3 : It may be noted that in [122, Lemma 2.2.1]
the conclusion (6.2.5) of lemma has been proved by assuming
an additional condition namely

$$\frac{\partial^3 |g(z)|}{\partial \Theta^3}\bigg|_{z=z_0} \geq 0.$$

LEMMA 6.2.2 : Let φ be a univalent mapping of $\bar{U}$ onto $\bar{\Omega}$,
$\varphi(0) = a$ and such that φ is analytic in $\bar{U}$ except for atmost
one pole on ∂U. Denote by $\eta(w)$ the argument of the outer
normal to $\partial \Omega$ at a finite boundary point $w \in \partial \Omega$. Let w
defined by $w(z) = a + w_n z^n + w_{n+1} z^{n+1} + \ldots$ be analytic in
U, with $w(z) \neq a$ and $n \geq 1$ and set $g(z) = \varphi^{-1}(w(z))$. Suppose
that there exists a point $z_0 = r_0 e^{i\Theta_0} \in U$ such that
$w_0 = w(z_0) \in \partial \Omega$ and $w(|z| < r_0) \subset \Omega$: If $\zeta_0 = \varphi^{-1}(w_0)$ then
there is a real number m, $m \geq n \geq 1$, such that

(6.2.9) $\text{arg }(z_0 w'(z_0)) = \text{arg }(\zeta_0 \varphi'(\zeta_0)) = \eta(w_0),$

(6.2.10) $|z_0 w'(z_0)| = m |\zeta_0 \varphi'(\zeta_0)| > 0,$

$$(6.2.11) \qquad \text{Re} \ \{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\} \geqq m \ \text{Re} \ \{1 + \frac{\zeta_0 \ \varphi''(\zeta_0)}{\varphi'(\zeta_0)}\} \ .$$

Further if,

$$(6.2.12) \quad \text{Im} \ [\frac{\zeta_0 \ \varphi''(\zeta_0)}{\varphi'(\zeta_0)}] . \text{Im}[1 + \frac{z_0 g''(z_0)}{g'(z_0)}]$$

$$\leqq \ \text{Re} \ [\frac{\zeta_0 \ \varphi''(\zeta_0)}{\varphi'(\zeta_0)}] \ \text{Re} \ [1 + \frac{z_0 g''(z_0)}{g'(z_0)} - m]$$

or equivalently

$$\text{Re} \ [\frac{\zeta_0 \ \varphi''(\zeta_0)}{\varphi'(\zeta_0)}(1 + \frac{z_0 w''(z_0)}{w'(z_0)} - m(1 + \frac{\zeta_0 \ \varphi''(\zeta_0)}{\varphi'(\zeta_0)}))] \geqq 0$$

and

$$(6.2.13) \qquad \frac{\partial^3 \ \arg \ (g(z))}{\partial \Theta^3} \Big|_{z=z_0} \leqq 0,$$

where $g(z) = \varphi^{-1}(w(z))$ and $\zeta = g(z)$, then

$$(6.2.14) \quad \text{Re} \ \{\frac{z_0 w'(z_0) + 3z_0^2 w''(z_0) + z_0^3 w'''(z_0)}{z_0 w'(z_0)}\}$$

$$\geqq m^2 \ \text{Re} \ \{\frac{\zeta_0 \ \varphi'(\zeta_0) + 3\zeta_0^2 \ \varphi''(\zeta_0) + \zeta_0^3 \ \varphi'''(\zeta_0)}{\zeta_0 \ \varphi'(\zeta_0)}\} \ .$$

Proof : The relations (6.2.9) to (6.2.11) have been obtained in [74]. However, we reproduce their proof for the sake of completeness and for the proof of (6.2.14) we follow the method similar to that of Miller and Mocanu [74].

Since w_0 is finite and φ is univalent at ζ_0, we have $\varphi'(\zeta_0) \neq 0$ and $\eta(w_0) = \eta(\varphi(\zeta_0)) = \arg(\zeta_0 \varphi'(\zeta_0))$. The function g given by $g(z) = \varphi^{-1}(w(z))$ is analytic in $|z| \leq |z_0|$ and satisfies $|g(z_0)| = 1$, $g(0) = 0$, and $|g(z)| \leq 1$ for $|z| \leq r_0$. Then g satisfies the conditions of Lemma 6.2.1. Since $w(z) = \varphi(g(z))$, $(\zeta = g(z))$, a simple computation yields

$$(6.2.15) \qquad zw'(z) = \zeta \varphi'(\zeta) \frac{zg'(z)}{g(z)}$$

$$(6.2.16) \quad \frac{z(zw'(z))'}{zw'(z)} = \frac{\zeta \varphi''(\zeta)}{\varphi'(\zeta)} \frac{zg'(z)}{g(z)} + \frac{z(zg'(z))'}{zg'(z)}$$

$$(6.2.17) \quad \frac{z(z(zw'(z))')'}{zw'(z)} = \frac{\zeta^3 \varphi'''(\zeta)}{\varphi'(\zeta)} \left(\frac{zg'(z)}{g(z)}\right)^2 + \frac{3\zeta^2 \varphi''(\zeta)}{\zeta \varphi'(\zeta)} \times$$

$$\left[\frac{z(zg'(z))'}{zg'(z)} \cdot \frac{zg'(z)}{g(z)}\right] + \frac{z(z(zg'(z))')'}{zg'(z)}.$$

Since $g(z)$ satisfies the condition (6.2.1) of Lemma 6.2.1, the conclusions (6.2.9) to (6.2.11) easily follows from (6.2.2) and (6.2.3) by using (6.2.15) and (6.2.16). Using (6.2.9) to (6.2.11), (6.2.17) gives

$$\mathrm{Re} \left\{ \frac{z_0 w'(z_0) + 3z_0^2 w''(z_0) + z_0^3 w'''(z_0)}{z_0 w'(z_0)} \right\}$$

$$= m^2 \mathrm{Re} \left\{ \frac{\zeta_0^3 \varphi'''(\zeta_0)}{\zeta_0 \varphi'(\zeta_0)} \right\} + 3m \, \mathrm{Re} \left\{ \frac{\zeta_0^2 \varphi''(\zeta_0)}{\zeta_0 \varphi'(\zeta_0)} \left(1 + \frac{z_0 g''(z_0)}{g'(z_0)}\right) \right\}$$

$$+ \mathrm{Re} \left\{ \frac{z_0 g'(z_0) + 3z_0^2 g''(z_0) + z_0^3 g'''(z_0)}{z_0 g'(z_0)} \right\}.$$

Using (6.2.12), (6.2.13) and (6.2.5) we get (6.2.14).

This completes the proof of Lemma 6.2.2.

REMARK 6.2.4 : For convenience one can write (6.2.12)
equivalently in the form either

$$(6.2.18) \quad \text{Re} \left\{ \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)} \left(1 + \frac{z_0 g''(z_0)}{g'(z_0)} - m \right) \right\} \geq 0$$

$$\text{or} \quad \text{Re} \left\{ \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)} \left(1 + \frac{z w''(z_0)}{w'(z_0)} - m \left(1 + \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)} \right) \right) \right\} \geq 0.$$

REMARK 6.2.5 : In [122, Lemma 2.2.2], the conclusion (6.2.14)
has been proved by replacing (6.2.12) by

$$\text{Im} \left[\zeta_0 \frac{\varphi''(\zeta_0)}{\varphi'(\zeta_0)} \right] \cdot \text{Im} \left[1 + \frac{z_0 g''(z_0)}{g'(z_0)} \right] \leq 0$$

and with an additional condition namely

$$\left. \frac{\partial^3 |g(z)|}{\partial \theta^3} \right|_{z = z_0} \geq 0, \quad (g(z) = \varphi^{-1}(w(z)) \text{ and } \zeta = g(z)).$$

We now proceed to prove our main theorems.

THEOREM 6.2.1 : Let w defined by $w(z) = a + w_n z^n + w_{n+1} z^{n+1} + \ldots$
be analytic in U with $w(z) \not\equiv a$ and $n \geq 1$, and let $z_0 \neq 0$
be a point of U such that

$$(6.2.19) \qquad |w(z_0)| = \max_{|z| \leq |z_0|} |w(z)|.$$

Then there is a real number M,

$$M \geqq n \frac{|w_0-a|^2}{|w_0|^2-|a|^2} \geqq n \frac{|w_0|-|a|}{|w_0|+|a|} \; ,$$

such that

$$(6.2.20) \qquad \frac{z_0 w'(z_0)}{w(z_0)} = M,$$

$$(6.2.21) \qquad \mathrm{Re} \; \{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\} \geqq M.$$

Further if,

$$(6.2.22) \quad \mathrm{Re} \; \{\bar{a}w_0(\bar{a}-\bar{w}_0)^2\} \leqq -2(\mathrm{Im}(\bar{a}w_0))^2,$$

$$(6.2.23) \quad 0 \leqq \mathrm{Re}[(|a|^2-\bar{a}w_0) \; \{(1 + \frac{z_0 w''(z_0)}{w'(z_0)} - m)(|w_0|^2-|a|^2)$$

$$- 2m(|a|^2-\bar{a}w_0)\}]$$

and

$$(6.2.24) \quad \frac{\partial^3}{\partial \theta^3} \arg \zeta \bigg|_{\zeta=\zeta_0} \leqq 0 \; \underline{\mathrm{where}} \; \; \zeta = w_0(\frac{a-w}{w\bar{a}-|w_0|^2}),$$

$$\zeta_0 = \frac{a-w_0}{\bar{a}-\bar{w}_0} \; ,$$

then

$$(6.2.25) \quad \mathrm{Re} \; \{\frac{z_0 w'(z_0)+3z_0^2 w''(z_0)+z_0^3 w''}{z_0 w'(z_0)}\} \geqq M^2.$$

Proof : If we let $\Omega = \{w: |w| \leqq |w_0|\}$, where $w(z_0) = w_0$, then $\varphi(\zeta) = w_0(\bar{w}_0 \zeta + a)/(w_0 + \bar{a}\zeta)$ is a conformal mapping of U onto Ω with $\varphi(0) = a$. Since $\zeta_0 = \varphi^{-1}(w_0)$, a simple calculation yields

$$(6.2.26) \qquad \zeta_0 = \frac{a - w_0}{\bar{a} - \bar{w}_0}$$

$$(6.2.27) \qquad \zeta_0 \varphi'(\zeta_0) = \frac{w_0 |w_0 - a|^2}{|w_0|^2 - |a|^2}$$

$$(6.2.28) \qquad \varphi'(\zeta) = \frac{w_0(|w_0|^2 - |a|^2)}{(w_0 + \bar{a}\zeta)^2} \quad \text{and}$$

$$(6.2.29) \qquad 1 + \frac{\zeta \varphi''(\zeta)}{\varphi'(\zeta)} = 1 - \frac{2\bar{a}\zeta}{w_0 + \bar{a}\zeta} = \frac{w_0 - \bar{a}\zeta}{w_0 + \bar{a}\zeta} \quad .$$

Therefore, by (6.2.26),

$$1 + \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)} = \frac{|w_0|^2 + |a|^2 - 2\bar{a}w_0}{|w_0|^2 - |a|^2}$$

and so

$$\text{Re } \left\{ 1 + \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)} \right\} = \frac{|w_0 - a|^2}{|w_0|^2 - |a|^2} \geqq \frac{(|w_0| - |a|)^2}{|w_0|^2 - |a|^2} = \frac{|w_0| - |a|}{|w_0| + |a|} \quad .$$

Differentiating (6.2.29) with respect to ζ and then multiplying by ζ we get

$$\frac{\zeta(\zeta(\zeta\varphi'(\zeta))')'}{\zeta\varphi'(\zeta)} = \left(\frac{\zeta(\zeta\varphi'(\zeta))'}{\zeta\varphi'(\zeta)}\right)^2 + \frac{(w_0 + \bar{a}\zeta)(-\bar{a}) - \bar{a}(w_0 - \bar{a}\zeta)}{(w_0 + \bar{a}\zeta)^2}$$

$$= \left(\frac{w_0 - \bar{a}\zeta}{w_0 + \bar{a}\zeta}\right)^2 - \frac{2\bar{a}w_0}{(w_0 + \bar{a}\zeta)^2} \quad .$$

Thus, by using (6.2.26),

$$(6.2.30) \quad \frac{\zeta_0^3 \varphi'''(\zeta_0) + 3\zeta_0^2 \varphi''(\zeta_0) + \zeta_0 \varphi'(\zeta_0)}{\zeta_0 \varphi'(\zeta_0)}$$

$$= \frac{-2\bar{a}w_0(\bar{a}-\bar{w}_0)^2 + [w_0(\bar{a}-\bar{w}_0) - \bar{a}(a-w_0)]^2}{[w_0(\bar{a}-\bar{w}_0) + \bar{a}(a-w_0)]^2}$$

$$= \frac{-2\bar{a}(\bar{a}-\bar{w}_0)^2 + (2\bar{a}w_0 - |w_0|^2 - |a|^2)^2}{(|a|^2 - |w_0|^2)^2} .$$

By Lemma 6.2.2, there exists a real number m satisfying $m \geq n \geq 1$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = m \; \zeta_0 \; \frac{\varphi'(\zeta_0)}{\varphi(\zeta_0)} = m \; \frac{|w_0 - a|^2}{|w_0|^2 - |a|^2} = M,$$

$$\mathrm{Re}\left[1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right] \geq m \; \mathrm{Re}\left[1 + \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)}\right] = m \; \frac{|w_0 - a|^2}{|w_0|^2 - |a|^2} = M.$$

Thus by (6.2.22), (6.2.23), (6.2.24) and (6.2.30) we get

$$\mathrm{Re}\left[\frac{z_0^3 w'''(z_0) + 3z_0^2 w''(z_0) + z_0 w'(z_0)}{z_0 w'(z_0)}\right]$$

$$\geq m^2 \; \mathrm{Re}\left[\frac{\zeta_0^3 \varphi'''(\zeta_0) + 3\zeta_0^2 \varphi''(\zeta_0) + \zeta_0 \varphi'(\zeta_0)}{\zeta_0 \varphi'(\zeta_0)}\right]$$

$$= m^2 \; \frac{|w_0 - a|^4}{(|w_0|^2 - |a|^2)^2} = M^2.$$

This completes the proof of the theorem.

REMARK 6.2.6 : Again in [122, Theorem 2.3.1], for the proof of (6.2.25) Sangeeta used an additional condition as remarked in Remark 6.2.5 and replaced (6.2.22) by

$$(6.2.31) \qquad \text{Re } [aw_o] \leq -2 \frac{(\text{Im } \bar{a}w_o)^2}{|a-w_o|^2} .$$

This shows our theorem improves the results obtained in [122].

We now use the above theorem to generate subclasses of bounded analytic functions and also to show that the third order complex differential equation have bounded solutions. In what follows $J > 0$, n will be a positive integer, and 'a' will be a complex number satisfying $|a| < J$. We set $\lambda \equiv \lambda(a,n,J) = \frac{n(J-|a|)}{J+|a|}$.

THEOREM 6.2.2 : Let Ω be a set in the complex plane and let $h(r,s,t,u) : \mathbb{C}^4 \longrightarrow \mathbb{C}$ be such that

(6.2.33) $h(r,s,t,u)$ is continuous in a domain $D \subset \mathbb{C}^4$

(6.2.34) $h(Je^{i\Theta},Ke^{i\Theta},L,M) \notin \Omega$ when $(Je^{i\Theta},Ke^{i\Theta},L,M) \in D$, $K \geq J\lambda$, $\text{Re}[Le^{-i\Theta}] \geq K(\lambda-1)$ and $\text{Re}[Me^{-i\Theta}+3Le^{-i\Theta}+K] \geq \lambda^2 K$. Let $w(z) = a+w_n z^n+w_{n+1}z^{n+1}+\ldots$ be analytic in U with $w(z) \not\equiv a$, $n \geq 1$ and at the point z_o where $|w(z)|$ attains its maximum value for $|z| \leq |z_o|$, let it satisfy (6.2.22) to (6.2.24). If $(w(z),zw'(z), z^2w''(z),z^3w'''(z)) \in D$ when $z \in U$ and

(6.2.35) $h(w(z),zw'(z),z^2w''(z),z^3w'''(z)) \in \Omega$ when $z \in U$

then $|w(z)| < J$ when $z \in U$.

Proof : It is clear that $|w(0)| = |a| < J$. Suppose that $z_0 = r_0 e^{i\theta_0} \in U$ $(0 < r_0 < 1)$ is a point of maximum for $|w(z)|$, i.e.,

(6.2.36) $\quad J = |w(z_0)| = \max_{|z| \leq r_0} |w(z)|,$

then by hypothesis the relations (6.2.22), (6.2.23) and (6.2.24) hold. From (6.2.36), $w(z_0) = Je^{i\theta}$ and since by (6.2.20) of Theorem 6.2.1

$$z_0 \frac{w'(z_0)}{w(z_0)} = M \geq \lambda,$$

we have $z_0 w'(z_0) = MJe^{i\theta} = Ke^{i\theta}$ where $K = MJ \geq J\lambda$. Also by (6.2.21) of Theorem 6.2.1, we have

$$\operatorname{Re} \{z_0 \frac{w''(z_0)}{w'(z_0)}\} > M-1 \geq \lambda-1$$

and this simplifies to

$$\operatorname{Re} \{\frac{z_0^2 w''(z_0)}{z_0 w'(z_0)}\} = \operatorname{Re} \{\frac{z_0^2 w''(z_0)}{Ke^{i\theta}}\} \geq \lambda-1,$$

or

$$\operatorname{Re} \{z_0^2 w''(z_0)e^{-i\theta}\} = \operatorname{Re}[L\, e^{-i\theta}] \geq K(\lambda-1)$$

with $L = z_0^2 w''(z_0)$.

Again by (6.2.25), we obtain

$$\operatorname{Re} \{\frac{z_0^3 w'''(z_0)+3z_0^2 w''(z_0)+z_0 w'(z_0)}{z_0 w'(z_0)}\} \geq \lambda^2$$

or $\qquad$ Re $[Me^{-i\theta} + 3Le^{-i\theta} + K] \geqq \lambda^2 K$

with $M = z_0^3 w'''(z_0)$.

Therefore at the point $z = z_0$, by (6.2.34) we obtain

$$(w(z_0), z_0 w'(z_0), z_0^2 w''(z_0), z_0^3 w'''(z_0)) \notin \Omega .$$

This contradicts (6.2.35) and hence we must have $|w(z)| < J$ for $z \in U$.

REMARK 6.2.7 : If we take $\Omega = \{\eta \in \mathbb{C} : |\eta| < 1\}$ in the above theorem we obtain the generalized and improved form of the results obtained in [122, Theorem 2.3.2].

EXAMPLE 6.2.1 : Let $h_1(r,s,t,u) = \frac{1}{2}[r+s+3t+u]$ with $D = \mathbb{C}^4$, $n = 1$, $a \to 0$ and $\Omega = \{\eta \in \mathbb{C} : |\eta| < J\}$. Condition (6.2.33) is satisfied and we need to show that

$$|Je^{-i\theta} + Ke^{-i\theta} + 3L + M| \geqq 2J, \text{or } |J+K+3Le^{-i\theta} + Me^{-i\theta}| \geqq 2J$$

when $K \geqq J$, $\text{Re}(Le^{-i\theta}) \geqq 0$ and $\text{Re}(Me^{-i\theta} + 3Le^{-i\theta}) \geqq 0$.

But this follows immediately since

$$|2^{-1}(J+K+3Le^{-i\theta} + Me^{-i\theta})| \geqq \frac{1}{2} \text{Re}(J+K+3Le^{-i\theta} + Me^{-i\theta}) \geqq \frac{J+J}{2} = J.$$

Hence if w defined by $w(z) = w_1 z + \ldots$ is analytic in U, with $w(z) \neq 0$, $0 < J$ satisfying

$$(6.2.38) \quad \frac{\partial^3}{\partial \theta^3} \arg(\zeta) \Big|_{\zeta = \zeta_0} \leqq 0, \text{where } \zeta = \frac{w(z)}{w(z_0)}, \zeta_0 = \frac{w(z_0)}{w_0(z_0)}, z = re^{i\theta},$$

$0 < r < 1$ and z_0 is the point of maximum for $|w(z)|$, then

$$|w(z)+zw'(z)+3z^2w''(z)+z^3w'''(z)| < 2J \text{ for } z \in U$$

implies $|w(z)| < J$ for $z \in U$.

EXAMPLE 6.2.2 : Let $h_2(r,s,t,u) = rs(r+pt+qu)$, $p \geq 3q$ and q is real and non-negative with $D = \mathbb{C}^4$, $a = 0$, $\lambda = 1$ (i.e. for $n = 1$), and $\Omega = \{\eta \in \mathbb{C} : |\eta| < J^3\}$, condition (6.2.33) is satisfied and we only need to check (6.2.34). Thus for, $K \geq J > 0$, $\text{Re}(Le^{-i\theta}) \geq 0$ and $\text{Re}(Me^{-i\theta}+3Le^{-i\theta}) \geq 0$,

$$
\begin{aligned}
|h_2(Je^{i\theta},Ke^{i\theta},L,M)| &= JK|J+pLe^{-i\theta} + qMe^{-i\theta}| \\
&= JK|J+q(3Le^{-i\theta}+Me^{-i\theta})+(p-3q)Le^{-i\theta}| \\
&\geq JK \text{ Re } \{J+q(3Le^{-i\theta}+Me^{-i\theta})+(p-3q)Le^{-i\theta}\} \\
&\quad JK \{J + q \text{ Re}(3Le^{-i\theta} + Me^{-i\theta}) \\
&\quad\quad + (p-3q) \text{ Re}(Le^{-i\theta})\} \\
&\geq J^2K \geq J^3.
\end{aligned}
$$

Hence if $w(z) = w_1 z + \ldots$ is analytic in U, $w(z) \neq 0$, $0 < J$ satisfying (6.2.38) then

$$|zw(z)w'(z)[w(z)+pz^2w''(z)+qz^3w'''(z)]| < J^3, \; z \in U$$

implies $|w(z)| < J$ for $z \in U$

provided $q \geq 0$ and $p \geq 3q$.

EXAMPLE 6.2.3 : Let $h_3(r,s,t,u) = r+As+Bt+Cu$, $D = \mathbb{C}^4$, $n = 1$, $a = 0$, $A \geq 0$, $B \geq 3C$, and $C \geq 0$ with $\Omega = \{\eta: |\eta| > J(1+A)\}$,

$J > 0$. It is easy to check that the conditions of Theorem 6.2.2 are satisfied. Hence if w defined by $w(z) = w_1 z + \ldots$ is analytic in U, with $w(z) \neq 0$, satisfying (6.2.38), then

$$|w(z) + Azw'(z) + Bz^2 w''(z) + Cz^3 w'''(z)| < J(1+A) \text{ for } z \in U$$

$$\text{implies } |w(z)| < J \qquad \text{for } z \in U.$$

A use of Theorem 6.2.2 leads to the following theorem which shows that certain second order differential equations have bounded solutions.

THEOREM 6.2.3 : Let h satisfy the conditions of Theorem 6.2.2, b(z) be an analytic function and $\Omega = b(U)$. If the differential equation

$$h(w(z), zw'(z), z^2 w''(z), z^3 w'''(z)) = b(z), \ w(0) = a, \ z \in U$$

has a solution w analytic in U satisfying (6.2.22) to (6.2.24) then $|w(z)| < J$ in U.

If we apply the above theorem to the Example 6.2.1 we obtain that if the solution w of the differential equation

$$w(z) + zw'(z) + 3z^2 w''(z) + z^3 w'''(z) = 2b(z), \ (w(0) = 0)$$

satisfies (6.2.38) where $\Omega = b(U)$ then one must have

$$|w(z)| < 1, \quad z \in U.$$

One may similarly construct other examples involving h_2, h_3 etc. as defined in Examples 6.2.2 and 6.2.3 respectively.

6.3 DIFFERENTIAL SUBORDINATION CONDITIONS FOR FUNCTIONS WITH POSITIVE REAL PART :

We first state

THEOREM 6.3.1 : Let p given by $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \ldots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If $z_o = r_o e^{i\theta_o}$ $(0 < r_o < 1)$ and $k = \operatorname{Re} p(z_o) = \min\limits_{|z| \leq r_o} \operatorname{Re} p(z)$ then

$$(6.3.1) \quad z_o p'(z_o) \leq \frac{-n|a - p(z_o)|^2}{2 \operatorname{Re}(a - p(z_o))} \leq \frac{-n}{2} \operatorname{Re}(a + p(z_o)),$$

$$(6.3.2) \quad \operatorname{Re}\left\{1 + z_o \frac{p''(z_o)}{p'(z_o)}\right\} > 0, \text{ and}$$

$$(6.3.3) \quad \operatorname{Re}\{z_o^2 p''(z_o) + z_o p'(z_o)\} \leq 0.$$

Further if

$$(6.3.4) \quad mY^2 - BY - A \geq 0, \quad \frac{\partial^3}{\partial\theta^3} \arg\left(\varphi^{-1}(p(z))\right)\Big|_{z=z_o} \leq 0$$

where

$$\zeta = \varphi^{-1}(p(z)), \quad \zeta_o = \varphi^{-1}(p(z_o)), \quad Y = \operatorname{Im}\left(1 + \frac{\zeta_o \varphi''(\zeta_o)}{\varphi'(\zeta_o)}\right),$$

$$A = \operatorname{Re}\left(1 + \frac{z_o p''(z_o)}{p'(z_o)}\right), B = \operatorname{Im}\left(1 + \frac{z_o p''(z_o)}{p'(z_o)}\right)$$

and $\varphi(\zeta) = (a - (2k - \bar{a})\zeta)/(1 - \zeta)$, then

$$(6.3.5) \quad \operatorname{Re}\left\{\frac{z_o^3 p'''(z_o) + 3z_o^2 p''(z_o) + z_o p'(z_o)}{z_o p'(z_o)}\right\} \geq \frac{-m^2|a - p(z_o)|^2}{2(\operatorname{Re}(a - p(z_o)))^2}$$

or equivalently

$$\mathrm{Re}[z_0^3 p'''(z_0) + 3z_0 p''(z_0) + z_0 p'(z_0)] \leqq \frac{-m^2}{2} z_0 p'(z_0) \frac{|a-p(z_0)|^2}{2(\mathrm{Re}(a-p(z_0)))^2}$$

where $m \geqq n \geqq 1$.

Proof : If we let $k = \mathrm{Re}\, p(z_0)$ and $\Omega = \{w : \mathrm{Re}\, w \geqq k\}$ then

$$(6.3.6) \qquad \varphi(\zeta) = \frac{a-(2k-\bar{a})\zeta}{1-\zeta}$$

is a conformal mapping of U onto Ω with $\varphi(0) = a$. Setting $\zeta_0 = \varphi^{-1}(p(z_0))$ from (6.3.6) we obtain

$$(6.3.7) \qquad \zeta_0 = \frac{p(z_0) - a}{p(z_0) - (2k - \bar{a})}$$

$$(6.3.8) \qquad \zeta_0 \varphi'(\zeta_0) = \frac{-|a - p(z_0)|^2}{2\, \mathrm{Re}(1-p(z_0))} ,$$

$$(6.3.9) \qquad \mathrm{Re}\, \{1 + \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)}\} = 0,$$

and

$$(6.3.10) \quad \mathrm{Re}\, \{\frac{\zeta_0^3 \varphi'''(\zeta_0) + 3\zeta_0^2 \varphi''(\zeta_0) + \zeta_0 \varphi'(\zeta_0)}{\zeta_0 \varphi'(\zeta_0)}\}$$

$$= \frac{-|p(z_0) - a|^2}{2(\mathrm{Re}(a-p(z_0)))^2} ,$$

We now use Lemma 6.2.2 to complete the proof of this theorem. By (6.2.9) and (6.2.10) of Lemma 6.2.2, and (6.3.8) we see that $z_0 p'(z_0)$ must be a negative real number and that (6.3.1) is satisfied. By applying (6.2.9) to (6.2.11) we obtain

(6.3.2). We obtain (6.3.3) by multiplying (6.3.2) by the negative number $z_0 p'(z_0)$. Further by using (6.3.4), (6.3.10) and (6.2.14) we obtain (6.3.5). This completes the proof of our theorem.

We now use the above theorem to generate certain subclasses of functions with positive real part and also to show that certain third order differential equations have solutions with positive real part. We first describe the following class of generating functions.

DEFINITION 6.3.1 : Let $r = r_1 + ir_2$, $s = s_1 + is_2$, $t = t_1 + it_2$, $u = u_1 + iu_2$ and Ω be any set in the complex plane and a be a complex number satisfying Re $a > 0$. Suppose that m be a real number, n a positive integer with $m \geq n \geq 1$ and $\Psi_n(a, \Omega)$ the set of functions $\Psi(r, s, t, u): \mathbb{C}^4 \longrightarrow \mathbb{C}$ satisfying:

(6.3.11) $\Psi(r, s, t, u)$ is continuous in a domain D of $\mathbb{C}^4$

(6.3.12) $(a, 0, 0, 0) \in D$ and $\Psi(a, 0, 0, 0) \in \Omega$

(6.3.13) $\Psi(ir_2, s_1, t_1 + it_2, u_1 + u_{2i}) \notin \Omega$ when
$(r_2 i, s_1, t_1 + it_2, u_1 + u_2) \in D$,

$$s_1 \leq - \frac{n|a - ir_2|^2}{2 \, \text{Re } a}, \quad s_1 + t_1 \leq 0$$

and
$$u_1 + 3t_1 + s_1 \leq -m^2 \, \frac{s_1|a - ir_2|^2}{2(\text{Re } a)^2}.$$

We let $\Psi_n(\Omega) \equiv \Psi_n(1, \Omega)$ and $\Psi(\Omega) = \Psi_1(1, \Omega)$.

DEFINITION 6.3.2 : The function q defined by

$$q(z) = a + q_n z^n + q_{n+1} z^{n+1} + \ldots, \quad q(z) \not\equiv a,$$

analytic in U, is said to be in the class $\tilde{R}_n(\varphi)$ if there exists a point $z_0 = r_0 e^{i\theta_0} \in U$, $0 < r_0 < 1$ such that

$$0 = \operatorname{Re} q(z_0) = \min_{|z| < r_0} \operatorname{Re} q(z)$$

satisfying

$$mY^2 - BY - A \geq 0$$

$$\left. \frac{\partial^3}{\partial \theta^3} \arg \varphi^{-1}(q(z)) \right|_{z=z_0} \leq 0$$

where $\zeta = \varphi^{-1}(q(z))$, $\zeta_0 = \varphi^{-1}(q(z_0))$, $Y = \operatorname{Im}(1 + \frac{\zeta_0 \varphi''(\zeta_0)}{\varphi'(\zeta_0)})$

$A = \operatorname{Re}(1 + \frac{z_0 q''(z_0)}{q'(z_0)}$, $B = \operatorname{Im}(1 + \frac{z_0 q''(z_0)}{q'(z_0)})$ and

$\varphi(\zeta) = (a + \bar{a}\zeta)/(1-\zeta)$.

THEOREM 6.3.2 : Let $\Psi \in \Psi_n(a, \Omega)$ with corresponding domain D and let $p \in \tilde{R}_n(\varphi)$. If $(p(z), zp'(z), z^2 p''(z), z^3 p'''(z)) \in D$ when $z \in U$ and

(6.3.14) $\{\Psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z)) : |z| < 1\} \subset \Omega$

then $\operatorname{Re} p(z) > 0$ for $z \in U$.

Proof : Since $p \in \tilde{R}_n(\varphi)$, there exists a point $z_0 = r_0 e^{i\theta_0} \in U$, $0 < r_0 < 1$ such that

$$0 = \operatorname{Re} p(z_0) = \min_{|z| \leq r_0} \operatorname{Re} p(z).$$

Applying Theorem 6.3.1 we obtain

$$z_o p'(z_o) \leqq -n \frac{|a - \text{Im } p(z_o)|^2}{2 \text{ Re } a} \ ,$$

$$\text{Re } \{z_o^2 p''(z_o) + z_o p'(z_o)\} \leqq 0$$

and

$$\text{Re } \{z_o^3 p'''(z_o) + 3z_o^2 p''(z_o) + z_o p'(z_o)\}$$

$$\leqq -m^2 z_o p'(z_o) \frac{|a - \text{Im } p(z_o)|^2}{2(\text{Re } (a))^2} \ .$$

Using (6.3.13) and the fact that $\Psi \in \Psi_n(a, \Omega)$ we must have

$$\Psi(p(z_o), z_o p'(z_o), z_o^2 p''(z_o), z_o^3 p'''(z_o)) \notin \Omega.$$

But this contradicts (6.3.14) and so we must have $\text{Re } p(z) > 0$ for all $z \in U$.

REMARK 6.3.1 : Note that the condition (6.3.14) is not vacuous concept : $p(z) = a + p_n z^n$ will satisfy (6.3.14) for small $|p_n|$.

EXAMPLE 6.3.1 : Let

$$\Psi_1(r, s, t, u) = r + ks + m^2 \frac{s}{2}(1 - r^2) + 4t + u, \ n = 1$$

$m \geq 1$, $k \geq 2$, $\Omega = \{\eta \in \mathbb{C} : \text{Re } \eta > -(k-2)/2\}$ and $a = 1$. Then Ψ_1 is continuous in the domain $D = \mathbb{C}^4$, $(1, 0, 0, 0) \in D$ and $\Psi_1(1, 0, 0, 0) = 1 \in \Omega$.

Further for $s_1 \leqq \frac{-1}{2}(1 + r_2^2)$, $s_1 + t_1 \leqq 0$ and

$$s_1 + 3t + u_1 \leqq -m^2 \frac{s_1(1 + r_2^2)}{2} \quad \text{we have}$$

$$\text{Re } \Psi_1(ir_2, s_1, t_1+it_2, u_1+iu_2)$$

$$= ks_1 + m^2 \frac{s_1}{2}(1+r_2^2) + 4t_1+u_1$$

$$= (k-2)r_1+s_1+t_1 + [s_1+m^2 \frac{s_1(1+r_2^2)}{2} + 3t_1+u_1]$$

$$\leqq \frac{-(k-2)}{2}$$

i.e. $\Psi_1(ir_2, s_1, t_{1+i}t_2, u_1+iu_2) \notin \Omega$.

This shows that $\Psi_1 \in \Psi(1,\Omega)$. Now applying Theorem 6.3.2 to Ψ_1 we obtain that for $p \in \tilde{R}_1(\varphi)$,

$$\text{Re } \{p(z)+kzp'(z) + m^2 \frac{zp'(z)}{2}(1-p^2(z))+4z^2p''(z)+z^3p'''(z)\}$$

$$> \frac{-(k-2)}{2}, z \in U$$

implies Re $p(z) > 0$ in U.

The above theorem also has an interpretation in terms of differential equations as given in the following theorem. The proof will not be presented as it follows immediately from Theorem 6.3.2.

THEOREM 6.3.3 : Let q be analytic function satisfying $q(0) = a$ with Re $a > 0$, $q(U) = \Omega$ and $\Psi \in \Psi_n(a,\Omega)$. If the differential equation

$$\Psi(p(z), zp'(z), z^2p''(z), z^3p'''(z)) = q(z), (p(0) = a),$$

has a solution p analytic in U such that $p \in \tilde{R}_n(\varphi)$ then Re $p(z) > 0$ in U.

CHAPTER - VII

ON SOME SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS

7.1 INTRODUCTION :

In previous chapters we have studied a few subclasses of univalent analytic functions in U. In this chapter, we study certain subclasses of functions of the form

$$(7.1.1) \qquad g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$$

which are univalent and analytic in $0 < |z| < 1$. Thus, at $z = 0$, these functions have a simple pole. Denote this class by Σ.

Let $\Sigma^*(A,B)$, $\Sigma_K(A,B)$ $(-1 \leq B < 1, B < A)$ be the subclasses of functions in Σ satisfying

$$(7.1.2) \qquad \frac{-zg'(z)}{g(z)} \prec \frac{1+Az}{1+Bz} \ , \ z \in U$$

and

$$(7.1.3) \qquad -\left(\frac{zg''(z)}{g'(z)} + 1\right) \prec \frac{1+Az}{1+Bz} \ , \ z \in U$$

respectively. $\Sigma^*(1-2\beta,-1)$ and $\Sigma_K(1-2\beta,-1)$ $(0 \leq \beta < 1)$ are respectively the well-known subclasses of Σ consisting of functions meromorphic starlike and meromorphic convex of order β. Denote by $\Sigma^*(1-2\beta,-1) = \Sigma^*(\beta)$ and $\Sigma^*(0) = \Sigma^*$; $\Sigma_K(1-2\beta,-1) = \Sigma_K(\beta)$ and $\Sigma_K(0) = \Sigma_K$.

We say that a function $g \in \Sigma$ is meromorphic λ-spirallike of order β in $0 < |z| < 1$ if

$$(7.1.4) \qquad -\mathrm{Re} \ \{e^{i\lambda} \frac{zg'(z)}{g(z)}\} > \beta \cos \lambda \ , \ z \in U$$

for $0 \le \beta < 1$ and $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$. We denote this class by $\Sigma^{*\lambda}(\beta)$.

Similarly we say that a function Ψ of the form (7.1.1) analytic in $0 < |z| < 1$ is meromorphic class to convex of order β and type λ $(\beta < -1/d_{-1}, \ 0 \le \lambda < 1)$ in $0 < |z| < 1$ if there exists a function φ of the form

$$(7.1.5) \qquad \varphi(z) = \frac{d_{-1}}{z} + \sum_{n=0}^{\infty} d_n z^n, \quad d_{-1} \ne 0,$$

which is meromorphic starlike of order λ (i.e. φ need not be normalized) such that

$$(7.1.6) \qquad \mathrm{Re} \ \{\frac{z \ \Psi'(z)}{\varphi(z)}\} > \beta, \ z \in U.$$

If we choose $\varphi(z) = -\frac{1}{z}$, then the condition of meromorphic close-to-convexity reduces to

$$(7.1.7) \qquad -\mathrm{Re} \ \{z^2 \ \Psi'(z) \} > \beta, \ z \in U.$$

Now let g and Ψ be two functions with series expansions

$$g(z) = \frac{d_{-1}}{z} + \sum_{n=0}^{\infty} d_n z^n, \quad (d_{-1} \ne 0)$$

$$\Psi(z) = \frac{e_{-1}}{z} + \sum_{n=0}^{\infty} e_n z^n, \quad (e_{-1} \ne 0)$$

which are analytic and univalent in $0 < |z| < 1$. In [109],
Robertson showed that the convolution or Hadamard product
$g * \Psi$ of such functions defined by

$$(g * \psi)(z) = \frac{d_{-1}e_{-1}}{z} + \sum_{n=0}^{\infty} d_n e_n z^n$$

is analytic and univalent in $0 < |z| < 1$ (for $d_{-1} = e_{-1} = 1$).
Furthermore he proved that $(g * \Psi)$ is not only close-to-
convex of order 0 in $0 < |z| < 1$ with respect to $\varphi(z) = -1/z$
but also meromorphic starlike of order 0 in $0 < |z| < 1$.

To a certain extent the work on univalent meromorphic
functions has paralleled that of univalent analytic functions
since one is tempted to search for a class of functions in Σ
which is analogous to that of analytic case. However, though
a large number of papers have appeared dealing with the
classes defined, through convolution, of analytic functions,
it is somewhat surprising that no attempt appears to have
been made in defining the classes, through convolution,
of meromorphic functions. In the present chapter we make a
modest attempt in this direction.

In Section 7.2, we give a necessary and sufficient
condition for a function g in Σ to be in $\Sigma^*(A,B)$ and
$\Sigma_K(A,B)$ $(-1 \leqq \dot{B} < A \leqq 1)$ respectively in terms of
convolution. In Section 7.3, we give containment relation
for the classes $T_{\delta+1}^M(A,B)$. In Section 7.4, we study certain
integral transforms in the class $T_\delta^M(A,B)$ and in turn

to the classes $\Sigma^*(A,B)$ and $\Sigma_K(A,B)$ which are much more general than the one considered by Goel and Sohi [37] and Bajpai [4]. In the last section of this chapter we study some sort of converse problem for functions in $T^M(1-2\beta,-1)$.

7.2 CONVOLUTION THEOREMS :

In this section we use convolution techniques to obtain necessary and sufficient condition for a function $g \in \Sigma$ to be in $\Sigma^*(A,B)$.

THEOREM 7.2.1 :   A function g in Σ is in $\Sigma^*(A,B)(-1 \leq B < A \leq 1)$ if and only if

$$(7.2.1) \quad [g(z) * \{\frac{1 + (1-(A-2B)x)(A-B)^{-1}x^{-1}z}{z(1-z)^2}\}] \neq 0$$

for $0 < |z| < 1$, $|x| = 1$.

Proof :   The function g in Σ is in $\Sigma^*(A,B)(-1 \leq B < A \leq 1)$ if and only if

$$(7.2.2) \qquad -\frac{zg'(z)}{g(z)} \neq \frac{1+Ax}{1+Bx}$$

for $z \in U$ and $|x| = 1$.

Since $-(zg'(z)/g(z)) = 1$ at $z = 0$, (7.2.2) is equivalent to

$$(7.2.3) \quad -zg'(z)(1+Bx) - g(z)(1+Ax) \neq 0, \quad 0 < |z| < 1.$$

Since $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n = g(z) * (\frac{1}{z(1-z)})$

$$-zg'(z) = \frac{1}{z} - \sum_{n=0}^{\infty} n\, b_n z^n$$

$$= g(z) * (\frac{1}{z} - \frac{z}{(1-z)^2}), \quad 0 < |z| < 1$$

$$= g(z) * (-\frac{1-2z}{z(1-z)^2})$$

so that the left hand side of (7.2.3) may be expressed as

$$g(z) * [\frac{1 + (1-(A-2B)x) (A-B)^{-1}x^{-1}z}{z(1-z)^2}] \neq 0$$

for $0 < |z| < 1$, $|x| = 1$, which is the desired convolution condition .

If we set $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, then

$$(zg'(z))' = (-zg'(z)) * (-\frac{1-2z}{z(1-z)^2})$$

and so from Theorem 7.2.1, and the identity

$$zg'(z) * \Psi(z) = g(z) * z \Psi'(z), \quad g, \Psi \in \Sigma$$

we obtain

THEOREM 7.2.2 : A function g in Σ is in $\Sigma_K(A,B)(-1 \leqq B < A \leqq 1)$ if and only if

$$g(z) * [\frac{-1+3(1-(A-2B)x)(A-B)^{-1}x^{-1}z+2(1-(A-2B)x)(A-B)^{-1}x^{-1}z^2}{z(1-z)^3}] \neq 0$$

for $0 < |z| < 1$, $|x| = 1$.

REMARK 7.2.2 :   For $A = 1-2\beta$, $B = -1$, Theorems 7.2.1 and 7.2.2 give necessary and sufficient convolution conditions for a function $g \in \Sigma$ to be in $\Sigma^*(\beta)$ and $\Sigma_K(\beta)(0 \leq \beta < 1)$

respectively.

THEOREM 7.2.3 : A function $g \in \Sigma$ is λ-spiral-like of order β $(0 \leq \beta < 1)$ in $0 < |z| < 1$, if and only if

$$(7.2.4) \quad g(z) * \left[\frac{1 + \left(\dfrac{2\beta \cos \lambda \ e^{-i\lambda} - e^{-2i\lambda} - 2 - x}{1 + e^{-2i\lambda} - 2\beta \cos \lambda \ e^{-i\lambda}} \right) z}{z(1-z)^2} \right] \neq 0$$

for $0 < |z| < 1$, $|x| = 1$.

Proof : The function $g \in \Sigma$ is λ-spiral-like of order β $(0 \leq \beta < 1)$ if and only if

$$(7.2.5) \quad -\mathrm{Re} \left\{ e^{i\lambda} \frac{zg'(z)}{g(z)} \right\} > \beta \cos \lambda, \ z \in U, \ |\lambda| < \frac{\pi}{2}.$$

Since $\dfrac{-e^{i\lambda} \dfrac{zg'(z)}{g(z)} - i \sin \lambda}{\cos \lambda} = 1$ at $z = 0$,

(7.2.5) is equivalent to

$$\frac{-e^{i\lambda} \dfrac{zg'(z)}{g(z)} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + (2\beta - 1)x}{1 + x}, z \in U, |x| = 1, \ x \neq -1$$

which simplifies to

$$(7.2.6) \quad -(1 + \bar{x})zg'(z) + (e^{-2i\lambda} - 2\beta \cos \lambda \ e^{-i\lambda} - \bar{x}) g(z) \neq 0,$$

where $\bar{x}$ stands for conjugate of x.

The remainder of the argument is the same as that of Theorem 7.2.1. Hence the theorem.

7.3. CONTAINMENT RELATION FOR A NEW SUBCLASS OF Σ :

Now we define the class $T_\delta^M(A,B)$ as follows :

DEFINITION 7.3.1 : Let δ, A, B be **arbitrarily fixed** real numbers such that $\delta > -1$, $-1 \leq B < 1$ with $B < A$. A function $g \in \Sigma$ is said to be in the class $T_\delta^M(A,B)$ if it satisfies

$$(7.3.1) \qquad -[\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2] \prec \frac{1+Az}{1+Bz} \ , \ z \in U,$$

where

$$(7.3.2) \qquad E^\delta g(z) = \frac{1}{z(1-z)^{\delta+1}} * g(z), \quad \delta > -1, \ 0 < |z| < 1.$$

It is readily seen that for $\delta = n \in N \cup \{0\}$

$$E^n g(z) = \frac{z^{-1}}{n!} \frac{d^n}{dz^n} (z^{n+1}g(z)).$$

With this notation the well-known condition for $g \in \Sigma$ to be in $\Sigma^*(A,B)$ can be written as

$$g \in \Sigma^*(A,B) \text{ if and only if } g \in T_o^M(A,B).$$

Note that $E^o g(z) = g(z)$, $E^1 g(z) = zg'(z) + 2g(z)$

and so $\quad E^1 g(z)-2E^o g(z) = zg'(z) = \frac{2z-1}{z(1-z)^2} * g(z).$

For the proof of our next theorem we need the following lemma :

LEMMA 7.3.1 : For $g \in \Sigma$ and $\delta > -1$ we have the following identity

$$(7.3.3) \quad z \frac{d}{dz}(E^\delta g(z)) = (\delta+1)E^{\delta+1}g(z)-(\delta+2)E^\delta g(z), \quad 0 < |z| < 1$$

where $E^\delta g(z)$ is given by (7.3.2).

<u>Proof</u> : If we set $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, then

$$(7.3.4) \quad E^\delta g(z) = \frac{1}{z(1-z)^{\delta+1}} * g(z)$$

$$\equiv [\frac{1}{z} + \sum_{n=0}^{\infty} \binom{\delta+n+1}{n+1}z^n] * [\frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n].$$

Now equating the coefficients of z^{-1}, constant term and z^n in the expansion of $(\delta+1) E^{\delta+1}g(z) - (\delta+2) E^\delta g(z)$ with the corresponding coefficients in the expansion of $z(E^\delta g(z))'$, the result follows.

<u>THEOREM 7.3.1</u> : <u>Let</u> $\delta > -1$.

(a) <u>Suppose that the constants A,B and</u> $\delta > -1$ <u>satisfy</u>

$$(7.3.5) \quad B < A \leqq \frac{\delta + 1 + B}{\delta + 2} \quad \underline{for} \ -1 \leqq B < 1.$$

<u>Then for</u> $g \in T^M_{\delta+1}(A,B)$ <u>we have</u>

$$(7.3.6) \qquad g \in T^M_\delta(A',B)$$

<u>where</u>

$$(7.3.7) \qquad A' = A + (A-B)/(\delta+1) . \quad \underline{Further,}$$

$$(7.3.8) \qquad -(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2) \prec - \frac{1}{\delta+1} [\frac{1}{Q(z)}]+2 \equiv \tilde{q}(z) \prec \frac{1+A'z}{1+Bz}, z \in$$

where

$$(7.3.9) \quad Q(z) = \begin{cases} \int_0^1 \left(\frac{1+Btz}{1+Bz}\right)^{-(\delta+2)\left(\frac{A-B}{B}\right)} t^\delta \, dt, & \underline{if} \ B \neq 0 \\[4mm] \int_0^1 \exp\{(\delta+2)(1-t)Az\} \, t^\delta \, dt, & \underline{if} \ B = 0. \end{cases}$$

(b) <u>Suppose</u> <u>that</u> <u>the</u> <u>constants</u> A,B, <u>and</u> $\delta > -1$ <u>satisfy</u>

$$B < A \leqq \min \left\{\frac{\delta+1+B}{\delta+2} \ , \ 2B\right\}$$

<u>with</u> $0 < B < 1$, <u>then</u>

$$(7.3.10) \qquad T_{\delta+1}^M(A,B) \subset T_\delta^M(1-2\rho \ ,-1)$$

where

$$(7.3.11) \quad \rho' = 2 - \left[F\left(1,(\delta+2)\left(\frac{A-B}{B}\right); \ \delta+2; \ B/(1+B)\right)\right]^{-1}$$

(c) <u>Suppose</u> <u>that</u> <u>for</u> $\delta > -1$

$$B < A \leqq \min \left\{\frac{\delta+1+B}{\delta+2} \ , \ 0\right\}$$

<u>with</u> $-1 \leqq B < 0$, <u>then</u>

$$(7.3.12) \qquad T_{\delta+1}^M(A,B) \subset T_\delta^M(1-2\rho'' \ ,-1)$$

where

$$(7.3.13) \quad \rho'' = 2 - \left[F\left(1,(\delta+2)\left(\frac{B-A}{B}\right); \delta+2; \ -B/(1-B)\right)\right]^{-1}$$

<u>and</u> $F(a,b;c;z)$ <u>is</u> <u>as</u> <u>defined</u> <u>by</u> (2.2.2). <u>The</u> <u>relations</u> (7.3.8), (7.3.10) <u>and</u> (7.3.12) <u>are</u> <u>all</u> <u>the</u> <u>best</u> **possible**

Proof : We follow the method similar to that of Theorem 2.2.1

Let $g \in T_{\delta+1}^{M}(A,B)$ where $\delta > -1$, $-1 \leq B < 1$ and $B < A$.

Set $\varphi(z) = z[zE^{\delta}g(z)]^{-1/(1+\delta)}$ and $r_1 = \sup \{r: \varphi(z) \neq 0, 0 < |z| < r < 1\}$.

Then φ is single valued in $0 < |z| < r_1$ and using (7.3.3)
it follows that p defined by

$$(7.3.14) \qquad p(z) = \frac{z\,\varphi'(z)}{\varphi(z)} = -\left(\frac{E^{\delta+1}g(z)}{E^{\delta}g(z)} - 2\right)$$

is analytic in $|z| < r_1$ and $p(0) = 1$. Since $g \in T_{\delta+1}^{M}(A,B)$,
(7.3.1) coupled with (7.3.14) and (7.3.3) easily leads to

$$-\left(\frac{E^{\delta+2}g(z)}{E^{\delta+1}g(z)} - 2\right) = \frac{1}{\delta+2} + \frac{\delta+1}{\delta+2}\left(p(z) + \frac{z\,p'(z)}{(\delta+1)(2-p(z))}\right) \prec \frac{1+Az}{1+Bz}, \; |z| < r.$$

In otherwords,

$$(7.3.15) \qquad P(z) + \frac{z\,P'(z)}{\beta\,P(z) + \nu} \prec \frac{1+Az}{1+Bz}$$

where

$$(7.3.16) \quad P(z) = \left(1 - \frac{1}{\delta+2}\right)p(z) + \frac{1}{\delta+2}, \; \beta = -(\delta+2), \; \nu = 2\delta + 3$$

It can be easily seen that for $-1 \leq B < 1$ and $A \neq B$

$$\cdot \; \text{Re} \left\{\beta\left(\frac{1+Az}{1+Bz}\right) + \nu\right\} > 0 \text{ in U iff A and B satisfy the}$$

following inequalities

$$\frac{-(\delta+1)+(2\delta+3)B}{\delta+2} \leq A \leq \frac{(\delta+1)+(2\delta+3)B}{\delta+2} \quad \text{for } -1 < B < 1$$

and $\qquad A \geqq - \dfrac{(3\delta+4)}{\delta+2}$ for B = -1.

It may be noted that (7.3.5) obviously satisfies the above inequalities and so it follows that

$$\text{Re } \{\beta(\tfrac{1+Az}{1+Bz}) + \nu\} > 0 \text{ in U under the condition}$$

(7.3.5). Using Lemma 2.2.1 we deduce that

(7.3.17) $\qquad P(z) \prec q(z) \prec \dfrac{1+Az}{1+Bz} , \quad |z| < r_1$

where q is the best dominant of (7.3.15) and is given by

$q(z) = \dfrac{\delta+1}{\delta+2} \tilde{q}(z) + \dfrac{1}{\delta+2}.$ Again by (7.3.19) we get

(7.3.18) $\quad p(z) \prec \tilde{q}(z) \equiv \dfrac{-1}{(\delta+1)}[\dfrac{1}{Q(z)}]+2 \prec \dfrac{1+(\dfrac{A(\delta+2)-B}{\delta+1}}{1+Bz}, |z| < r_1$

where Q(z) is given by (7.3.9). From (7.3.5) and (7.3.18) we see that Re p(z) > 0 in $|z| < r_1$. This, by (7.3.14), shows that φ is starlike (univalent) in $|z| < r_1$. Thus it is not possible that φ vanishes in $|z| < r_1$ if $r_1 < 1$. So we conclude that $r_1 = 1$. Therefore p is analytic in U. Hence by (7.3.14) and (7.3.18), $g \in T^M_{\delta+1}(A,B)$ implies

$-(\dfrac{E^{\delta+1}g(z)}{E^{\delta}g(z)} - 2) \prec \tilde{q}(z)$ provided δ , A and B satisfy (7.3.5).

This proves (7.3.6) and (7.3.8).

(b) Next we show that

(7.3.19) $\underset{|z|<1}{\text{Inf}} \{Re \, \tilde{q}(z)\} = \tilde{q}(1), \; \tilde{q}(z) = 2 - \dfrac{1}{\delta+1}[\dfrac{1}{Q(z)}]$

provided $\delta > -1$, A and B satisfy

$$(7.3.20) \qquad B < A < \min \{\frac{\delta+1+B}{\delta+2} , 2B\} .$$

If we set $a = -\beta (\frac{A-B}{B})$, $b = \beta + \nu$, $C = \beta+\nu+1$ ($\beta = -(\delta+2)$, $\nu = 2\delta+3$) then $c > b > 0$. From (7.3.9) by using (2.2.3), (2.2.4) and (2.2.5) we see that for $0 \neq B$

$$Q(z) = (1+Bz)^a \int_0^1 (1+Btz)^{-a} t^{b-1} dt$$

$$= \frac{\Gamma(b) \ \Gamma(c-b)}{\Gamma(c)} F(a,c-b;c;Bz/(1+Bz))$$

$$(7.3.21)$$

$$= \frac{\Gamma(b) \ \Gamma(c-b)}{\Gamma(c)} F(1,a;c;Bz/(1+Bz)).$$

Again (7.3.9), by (7.3.21) for $0 < B < 1$, $B < A < \min \{\frac{\delta+1+B}{\delta+2},2B\}$

(so that $c > a > 0$), can be rewritten as

$$Q(z) = \int_0^1 g(t,z) \ d\mu(t)$$

with

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz}$$

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a) \ \Gamma(c-a)} t^{a-1}(1-t)^{c-a-1} dt.$$

Using Lemma 2.2.2, (with $\lambda = 0$) and the method same as that of Theorem 2.2.1, we easily obtain

$$\text{Re} \ \{\frac{1}{Q(z)}\} \geqq \frac{1}{Q(1)} , z \in U.$$

This proves (7.3.21) and so by (7.3.8) we obtain (7.3.10)

(For the case $A = \min \{\frac{\delta+1+B}{\delta+2}, 2B\}$ we obtain (7.3.10) by

letting $A \longrightarrow [\min \{\frac{\delta+1+B}{\delta+2}, 2B\}]^+$).

(c) In a manner similar to that of part (b), using the

Lemma 2.2.2 (with $\lambda = \pi$) we can easily show

$$\inf_{|z|<1} \mathrm{Re}\ \tilde{q}(z) = \tilde{q}(-1)$$

provided $\delta > -1$, A and B satisfy $-1 \leq B < 0$ with

$B < A \leq \min \{0, \frac{(\delta+1)+B}{\delta+2}\}$. The proof of part (d) follows on

the same lines. Sharpness follows from the best dominant

property.

7.4   INTEGRAL TRANSFORMS :

We study in this section certain integral transforms

of functions in the class $T_\delta^M(A,B)$. For a function $g \in \Sigma$,

defined by $g(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n$, Bajpai [4] defined the

integral transform G_c by

$$(7.4.1) \qquad G_c(z) = \begin{cases} c \int_0^1 u^c g(uz)\,du \\[2ex] z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} b_n z^n \end{cases}$$

where c is real with $c \geq 1$ and showed that

$$(7.4.2) \qquad g \in \Sigma^*(\beta) \text{ implies } G_c \in \Sigma^*(\beta) \qquad (0 \leq \beta < 1).$$

Reddy and Juneja [105] showed that the relation (7.4.2) continues to hold if c in (7.4.1) is taken to be a complex number satisfying Re c > 0. They improved the relation (7.4.2) further by using a weaker condition on g. In the following theorem we give the sharp relation in generalized form as follows.

THEOREM 7.4.1 : Let δ > -1, c be a complex number satisfying Re c > 0 and the constants A, B, δ and c satisfy

$$(7.4.3) \quad B - \frac{(1-B) \text{ Re } c}{\delta+1} \leq A \leq B + \frac{(1+B) \text{ Re } c}{\delta+1} \quad \text{for } -1 < B < 1$$

and

$$(7.4.4) \quad A \geq -1 - \frac{2 \text{ Re } c}{\delta+1} \quad \text{for } B = -1.$$

Consider the integral transform G_c defined by (7.4.1).

(a) If $g \in T_\delta^M(A,B)$ then the function G_c defined by (7.4.1) satisfies $G_c \in T_\delta^M(A,B)$.

Furthermore

$$(7.4.5) \quad -[\frac{E^{\delta+1} G_c(z)}{E^\delta G_c(z)} - 2] \prec -\frac{1}{\delta+1}[\frac{1}{Q(z)}]+(\frac{c+\delta+1}{\delta+1}) \equiv \tilde{q}(z), z \in U$$

where

$$(7.4.6) \quad Q(z) = \begin{cases} \int_0^1 (\frac{1+Btz}{1+Bz})^{-(\delta+1)(\frac{A-B}{B})} t^{c-1} dt & \text{if } B \neq 0 \\ \\ \int_0^1 \exp\{-(1+\delta)A(t-1)z\} t^{c-1} dt & \text{if } B = 0. \end{cases}$$

(b) <u>If</u> c <u>is</u> <u>real</u> <u>with</u> c > 0, 0 < B < 1 <u>and</u> B < A $\leq$

min $\{B + \dfrac{(c+1)B}{\delta+1},\ B + \dfrac{(1+B)c}{\delta+1}\}$, <u>then</u> <u>for</u> g $\in$ $T_\delta^M(A,B)$

<u>we</u> <u>have</u>

(7.4.7) $G_c \in T_\delta^M(1-2\rho',-1)$

<u>where</u>

(7.4.8) $\rho' = \dfrac{1}{\delta+1} \{c+\delta+1-[F(1,(\delta+1)(\frac{A-B}{B});c+1;B/(1+B))]^{-1}\}$

(c) <u>If</u> c <u>is</u> <u>real</u> <u>with</u> c > 0,

$\qquad$ B < A $\leqq$ min $\{-\dfrac{B(c-\delta)}{\delta+1},\ B + \dfrac{(1+B)c}{\delta+1}\}$ <u>for</u> $-1 < B < 0$

and

$\qquad$ $-1 < A \leqq \dfrac{c-\delta}{\delta+1}$ <u>for</u> B = -1

<u>then</u> <u>for</u> g $\in$ $T_\delta^M(A,B)$ <u>we</u> <u>have</u>

(7.4.9) $G_c \in T_\delta^M(1-2\rho'',-1)$

<u>where</u>

(7.4.10) $\rho'' = \dfrac{1}{\delta+1} \{c+\delta+1-[F(1,(\delta+1)(\frac{B-A}{B});c+1;-B/(1-B))]^{-1}\}$.

<u>The</u> <u>results</u> <u>are</u> <u>all</u> <u>the</u> <u>best</u> <u>possible</u> <u>one</u>

<u>Proof</u> : Suppose that g $\in$ $T_\delta^M(A,B)$ and A,B,δ and c satisfy

(7.4.3) and (7.4.4).

Since G is defined by

$$G(z) \equiv G_c(z) = (z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n) * g(z)$$

and

$$E^\delta g(z) = z^{-1} + \sum_{n=0}^{\infty} \frac{\Gamma(n+\delta+2)}{\Gamma(\delta+1)} \frac{1}{(n+1)!} z^n$$

it can be easily seen that

(7.4.11) $\quad z \frac{d}{dz} [E^\delta G(z)] = c\, E^\delta g(z) - (1+c)\, E^\delta G(z)$

(7.4.12) $\quad z \frac{d}{dz} [E^\delta G(z)] = (\delta+1)\, E^{\delta+1}G(z) - (\delta+2)E^\delta G(z)$.

Equating (7.4.11) and (7.4.12) we get

(7.4.13) $\quad c\, E^\delta g(z) = (c-\delta-1)\, E^\delta G(z) + (\delta+1)\, E^{\delta+1}G(z).$

Let $\quad g \in T_\delta^M(A,B)$. We put

$$\varphi(z) = z[z\, E^\delta G(z)]^{-1/(\delta+1)}$$

and $r_1 = \sup \{r : \varphi(z) \neq 0,\ 0 < |z| < r < 1\}$. Then φ is singlevalued and analytic in $|z| < r_1$ and p defined by

(7.4.14) $\quad p(z) = \frac{z\, \varphi'(z)}{\varphi(z)} = -(\frac{E^{\delta+1}G(z)}{E^\delta G(z)} - 2)$

is analytic in $|z| < r_1$, $p(0) = 1$.
Using (7.4.11) and (7.4.12), (7.4.14) by differentiation reduces to

(7.4.15) $\quad -(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2) = p(z) + \frac{z p'(z)}{c+\delta+1-(\delta+1)p(z)},\ |z| < r_1.$

Since $g \in T_\delta^M(A,B)$, we have by (7.4.15) that

$$(7.4.16) \qquad p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \frac{1+Az}{1+Bz},$$

where $\qquad \beta = -(\delta+1)$, $\nu = c+\delta+1$.

Using Lemma 2.2.1 , we deduce that

$$(7.4.17) \quad p(z) \prec \tilde{q}(z) = \frac{1}{\beta}\left[\frac{1}{Q(z)}\right] - \frac{\nu}{\beta} \prec \frac{1+Az}{1+Bz}, \quad |z| < r_1.$$

It may be noted that for $\beta = -(\delta+1)$ and $\nu = c+\delta+1$ with $-1 \leqq B < 1$ and $B < A$

$$\mathrm{Re}\ \{\beta\ (\frac{1+Az}{1+Bz}) + \nu\} > 0 \text{ in } U$$

if the conditions (7.4.3) and (7.4.4) are satisfied.
Thus it is not possible that $\varphi(z)$ vanishes in $|z| = r_1$
if $r_1 < 1$ whenever A, B, δ and c satisfy (7.4.3) and (7.4.4).
So we conclude that $r_1 = 1$. Therefore p is analytic in U
and hence by (7.4.16) and (7.4.17) we obtain the first part
of the theorem from Lemma 2.2.1.

(b) For the second part it is enough to show that

$$(7.4.18) \quad \inf_{|z|<1} \tilde{q}(z) = \tilde{q}(1), \quad \tilde{q}(z) = \frac{-1}{\delta+1} \left[\frac{1}{Q(z)}\right] + \frac{c+\delta+1}{\delta+1}$$

where $Q(z)$ is given by (7.4.6), provided $\delta > -1$, c, A and B
satisfy $0 < B < 1$ with

$$A < B \leqq \min\ \{B + \frac{(c+1)B}{\delta+1}\ ,\ B + \frac{(1+B)c}{\delta+1}\}\ .$$

If we set $a = -\beta \frac{(A-B)}{B}$, $b = \beta + \nu$, $c' = \beta+\nu+1$

$(\beta = -(\delta+1)$, $\nu = c+\delta+1)$ then $c' > b > 0$. From (7.4.6) we

as before see that for $B \neq 0$

$$Q(z) = (1+Bz)^a \int_0^1 (1+Btz)^{-a} \, t^{b-1} \, dt$$

$$= \frac{\Gamma(b)\,\Gamma(c'-b)}{\Gamma(c')} \, F(1,a;c';Bz/(1+Bz)).$$

For $0 < B < 1$, $B < A < \min \{B + \frac{(c+1)B}{1+\delta}, B + \frac{(1+B)c}{1+\delta}\}$

we see that $c' > a > 0$ and so (7.4.6) can be rewritten as

$$Q(z) = \int_0^1 g(t,z) \, d\mu(t)$$

where

$$g(t,z) = \frac{1+Bz}{1+(1-t)Bz} \, ,$$

$$d\mu(t) = \frac{\Gamma(b)}{\Gamma(a)\,\Gamma(c'-a)} \, t^{a-1} \, (1-t)^{c'-a-1} \, dt \, .$$

Using Lemma 2.2.2, (with $\lambda = 0$) and the method similar to that of Theorem 2.2.1, we easily obtain

$$\text{Re} \, \{\frac{1}{Q(z)}\} \geq \frac{1}{Q(1)} \, , \, z \in U.$$

This proves (7.4.18) and hence by (7.4.17) and (7.4.14) we obtain (7.4.7). (For the case $A = \min \{B + \frac{(c+1)B}{1+\delta}, B + \frac{(1+B)c}{1+\delta}\}$ we obtain (7.4.7) by taking limit as in Theorem 7.3.1(b).

Part (c) can be proved on the similar lines using Lemma 2.2.2 with $\lambda = \pi$.

Taking $\delta = 0$ in the above theorem we obtain

COROLLARY 7.4.1 : Let c be a complex number satisfying Re c > 0, and consider

$$B - (1-B) \text{ Re } c < A \leq B + (1+B) \text{ Re } c \text{ when } -1 < B < 1,$$

and

$$A \geq -1 - 2 \text{ Re } c \text{ when } B = -1.$$

Define

$$G_c(z) = (\frac{1}{z} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n) * g(z) \text{ when Re } c > 0.$$

(a) If $g \in \Sigma^*(A,B)$ then the function G_c defined above satisfies

$$-\text{Re } \{z \frac{G_c'(z)}{G_c(z)}\} > \inf \text{ Re } [\cdot - [\frac{1}{Q(z)}] + {}^{c+\delta+1}]$$

where

$$Q(z) = \begin{cases} \int_0^1 (\frac{1+Btz}{1+Bz})^{\frac{B-A}{B}} t^{c-1} dt \text{ if } B \neq 0 \\ \\ \int_0^1 \exp \{A(1-t)z\} t^{c-1} dt \text{ if } B = 0. \end{cases}$$

(b) If c is real with c > 0, 0 < B < 1 and $B < A \leq \min \{B + (1+c)B, B + (1+B)c\}$ then

$$g \in \Sigma^*(A,B) \text{ implies } G_c \in \Sigma^*(1-2\rho_1', -1)$$

where $\quad \rho_1' = c+1 - [F(1, \frac{A-B}{B}; c+1; B/(1+B))]^{-1}.$

(c) If c is real with c > 0,

$$B < A \leq \min \{-Bc, B + (1+B)c\} \text{ when } -1 < B < 0$$

and

$$-1 < A \leq c \text{ when } B = -1$$

then

$$g \in \Sigma^*(A,B) \text{ implies } G_c \in \Sigma^*(1-\rho_2'',-1)$$

where

$$\rho_2'' = c+1 - [F(1, \frac{B-A}{B} ; c+1; -B/(1-B))]^{-1}.$$

REMARK 7.4.1 : Since $g \in \Sigma_K(A,B)$ if and only if $-zg'(z) \in \Sigma^*(A,B)$ one can show that the Corollary 7.4.1 remains true on replacing $\Sigma^*(A,B)$ by $\Sigma_K(A,B)$ both in the hypothesis and the conclusion of the theorem.

Substituting $A = 1-2\rho$ with $B = -1$ in part (c) of the above theorem, we get the following sharp result

COROLLARY 7.4.2 : Let $g \in \Sigma$, and for real $c > 0$, let

$$G_c(z) = (z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n) * g(z).$$

Then for $\frac{-(c-1)}{2} \leq \rho < 1$ we have

$$g \in \Sigma^*(\rho) \text{ implies } G_c \in \Sigma^*(\rho_3'')$$

and

$$g \in \Sigma^K(\rho) \text{ implies } G_c \in \Sigma^K(\rho_3'')$$

where $\rho_3'' = c+1 - [F(1, \frac{1-\rho}{2} ; c+1; 1/2)]^{-1}.$

The result is sharp.

The above corollary shows that the result obtained extend the earlier results of Bajpai [4], Goel and Sohi [37], Reddy and Juneja [105] and others.

7.5 INVERSE PROBLEM :

In the earlier section we considered the function G_c defined by

$$G_c(z) = [z^{-1} + \sum_{n=0}^{\infty} \frac{c}{c+n+1} z^n] * g(z).$$

If we solve the above equation for g in terms of G_c we see that

$$(7.5.1) \qquad g(z) = \frac{1}{c} [(1+c) G_c(z) + zG_c'(z)].$$

Given some property of $G_c(z)$, we can ask questions about the nature of $g(z)$. A problem of this type can be regarded as a sort of inverse problem.

For considering such an inverse problem of Theorem 7.4.1 for $A = 1-2\rho$ and $B = -1$, we need the special case of the following theorem.

THEOREM 7.5.1 : Let p and q be regular in U and Re $p(z) > 0$, $q(0) = 1$ and Re $q(z) > 0$ for $z \in U$. Further let $C(\neq 0)$ and D be complex constants such that $C+D \neq 0$. Then

$$(7.5.1) \quad \text{Re } \{p(z) + \frac{zp'(z)}{C+Dq(z)}\} > 0$$

$$\underline{in} \ |z| < \rho(C,D) = \frac{[|C|^2+2+4|D|+|D|^2 - \sqrt{R}]^{1/2}}{|C-D|}$$

$$(7.5.2) \qquad = \frac{|C+D|}{[|C|^2 + 2+4|D|+|D|^2 + \sqrt{R}]^{1/2}}$$

<u>where</u>

$$(7.5.3) \qquad R = (|C|^2+2+4|D|+|D|^2)^2 - |C^2-D^2|^2 \ .$$

<u>The</u> <u>result</u> <u>is</u> <u>sharp</u> <u>for</u> C <u>real</u> <u>and</u> <u>non-negative</u> <u>constant</u>.

<u>Proof</u> : Using the well-known inequalities [27]

$$(7.5.4) \qquad |p'(z)| \leq \frac{2 \ \mathrm{Re} \ p(z)}{1-r^2} \ , \quad |z| = r$$

and

$$(7.5.5) \qquad |q(z) - (\frac{1+r^2}{1-r^2})| \leq \frac{2r}{1-r} \ , \quad |z| = r,$$

we have

$$(7.5.6) \qquad \mathrm{Re} \ \{p(z) + \frac{zp'(z)}{C+Dp(z)}\} \geq \mathrm{Re} \ p(z) - |\frac{zp'(z)}{C+Dq(z)}|$$

$$\geq \mathrm{Re} \ p(z)(1 - \frac{1}{|C+Dq(z)|} \cdot \frac{2r}{1-r^2})$$

$$= \frac{\mathrm{Re} \ p(z)}{|C+Dq(z)|}(|C+Dq(z)| - \frac{2r}{1-r^2}).$$

Further we have,

$$|C+Dp(z)| - \frac{2r}{1-r^2} \geq |C + \frac{D(1+r^2)}{1-r^2}| - \frac{2r|D|}{1-r^2} - \frac{2r}{1-r^2}$$

$$= \frac{|C+D-(C-D)r^2| - 2(|D| + 1)r}{1-r^2} .$$

Since $|C+D - (C-D)r^2| - 2(|D|+1)r$ has positive value at $r = 0$ and negative value for $r = 1$ it has atleast one zero in $(0,1)$. Denoting its smallest positive root by $\rho(C,D)$ we get from (7.5.6) that

$$\text{Re } \{p(z) + \frac{zp'(z)}{C+Aq(z)}\} > 0 \text{ in } |z| < \rho(C,D).$$

It is easy to check that the smallest positive root of the equation

$$|C+D - (C-D)r^2| - 2(|D|+1)r = 0$$

is given by $\rho(C,D)$, where $\rho(C,D)$ and R are defined respectively by (7.5.2) and (7.5.3).

In [37], Goel and Sohi proved that if G_c belongs to Σ^* then g defined by (7.5.1) is meromorphically starlike in $0 < |z| < \sqrt{c/(c+2)}$. However their claim is incorrect. This can be easily seen from the following example.

$$\text{Let } G(z) = \frac{(1+z)^2}{z} = \frac{1}{z} + 2+z \text{ and } c = \frac{2}{3}$$

$$\text{Then } g(z) = \frac{1}{z} + 5 + 4z,$$

$$- \frac{zg'(z)}{g(z)} = \frac{1-4z^2}{1+5z+4z^2} = \frac{1-4z^2}{(4z+1)(z+1)}$$

and so according to Goel and Sohi g is meromorphically starlike in $0 < |z| < 1/2$. But this is not so because

$$- \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \Big|_{z = -\frac{1}{3}} = \frac{-5}{2} < 0.$$

The following theorem not only rectifies the result of Goel and Sohi [37] but also generalizes the corresponding results in this direction.

THEOREM 7.5.2 : Let $G_c \in T_\delta^M(\beta)$ $(0 \leq \beta < 1)$ and c be a complex number such that Re $c > 0$. Define g by

$$g(z) = \frac{1}{c} [(1+c)G_c(z) + zG_c'(z)].$$

Then g belongs to $T_\delta^M(\beta)$ in $|z| < \rho(C,D)$ for $C = c+(1+\delta)(1-\beta)$ and $D = -(1+\delta)(1-\beta)$ where $\rho(C,D)$ is given by (7.5.2).

Proof : Since $G_c \in T_\delta^M(\beta)$ we can write

$$p(z) = (1-\beta)^{-1} \left[-\left(\frac{E^{\delta+1}G_c(z)}{E^\delta G_c(z)} - 2 \right) - \beta \right]$$

where $p(0) = 1$ and Re $p(z) > 0$ for $z \in U$. As in the proof of Theorem 7.4.1 we get

$$(1-\beta)^{-1} \left\{ -\left(\frac{E^{\delta+1}g(z)}{E^\delta g(z)} - 2 \right) - \beta \right\} = p(z) + \frac{zp'(z)}{c+(\delta+1)(1-\beta)-(1+\delta)(1-\beta)p(z)} .$$

By hypothesis p is analytic, $Re\ p(z) > 0$ in U and $p(0) = 1$. Set

(7.5.7) $C = c+(1+\delta)(1-\beta)$ and $D = -(1+\delta)(1-\beta)$

so that $C+D \neq 0$.   Hence by Theorem 7.5.1 we obtain that

$$g \in T^M_\delta(\beta) \text{ in } |z| < \rho(C,D)$$

where $\rho(C,D)$ is obtained from (7.5.2) using (7.5.7). Substituting $\delta = 0$ in the above theorem and noting the Remark 7.4.1 we obtain the rectified form of [37] in the following corollary.

COROLLARY 7.5.1 :   If G_c belongs to $\Sigma^*(\beta)$ (or $\Sigma_K(\beta)$ resp.) for $0 \leq \beta < 1$ and

$$g(z) = \frac{1}{c} [(1+c)\ G_c(z) + zG'_c(z)], \quad Re\ c > 0$$

then

$$g \in \Sigma^*(\beta) \text{ (or } \Sigma_K(\beta) \text{ resp.) in } |z| < \rho(C,D)$$

with $C = c+(1-\beta)$ and $D = -(1-\beta)$.

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